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Numerical Estimation of Marcum's Q-Function using Monte Carlo Approximation Schemes

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ABSTRACT

The Marcum Q-Function is an important tool in the study of radar detection probabilities in Gaussian clutter and noise. Due to the fact that it is an intractable integral, much research has focused on finding good numerical approximations for it. Such approximations include numerical integration techniques, such as adaptive Simpson quadrature, and Taylor series approximations, induced by the modified Bessel function of order zero, which appears in the integrand. One technique which has not been explored in the literature is the sampling-based Monte Carlo approach. Part of the reason for this is that the integral representation of the Marcum Q-Function is not in the most suitable form for Monte Carlo methods. Using some recently derived techniques, we construct a number of sampling-based estimators of this function, and we consider their relative merits.

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EXECUTIVE SUMMARY

Radar detector performance and analysis are issues of paramount importance to the *modus operandi* of Electronic Warfare and Radar Division's Radar Modelling and Analysis Group. The research presented here is the practical extension of that which has appeared in a recent research report by the first author [DSTO-RR-0304, 'Stochastic Representations of the Marcum Q-Function and Associated Radar Detection Probabilities']. Hence it is in support of the ongoing long range research efforts for AIR 04/206. The purpose of this task is to provide the Royal Australian Air Force with technical advice on the performance of the Elta EL/M-2022 maritime radar, which is used in the AP-3C Orion fleet. Key performance measures of a radar include probabilities of false alarm and detection. The work presented here is concerned with the efficient estimation of a specific radar detection probability, known as Marcum's Q-function. This corresponds to the detection probability of a target in Gaussian clutter and noise, and so is a fundamental model in radar detection theory. This probability has been of interest to DSTO's research interests since the 1970s, through Task DST 74/130, which required the efficient estimation of the Marcum Q-Function.

In contrast to the techniques currently used in the radar literature, we investigate the application of Monte Carlo sampling methods to estimate this detection probability. Such methods have been investigated by the first author, in a number of DSTO reports, also in support of AIR 04/206 and its precursor AIR 01/217. The Marcum Q-Function does not *prima facie* suggest that Monte Carlo techniques would be suitable. New discoveries, through stochastic representations of the Marcum Q-function, have indicated that Monte Carlo techniques may be useful tools in the estimation of detection probabilities. We thus investigate whether these stochastic representations admit useful and efficient Monte Carlo estimators of the Marcum Q-Function.

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Glossary

Fundamental Symbols

$\mathbb{N}$ Natural numbers $\{0, 1, 2, \ldots\}$.

$\mathbb{R}$ Real numbers.

$\mathbb{R}^+$ Positive real numbers.

$\mathbb{P}$ Probability.

$\mathbb{E}$ Statistical expectation.

$\mathbf{V}$ Statistical variance.

$\mathbb{I}$ Indicator function: $\mathbb{I}[x \in A] = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{otherwise.} \end{cases}$

$:=$ Defined to be.

$\approx$ Approximately equal to.

$\stackrel{d}{=}$ Equality in distribution: $X \stackrel{d}{=} Y$ is equivalent to $\mathbb{P}(X \in A) = \mathbb{P}(Y \in A)$ for all sets A.

σ Signal to noise ratio (SNR).

τ Detection threshold.

$\widehat{\Xi}$ An estimator.

$a \wedge b$ Minimum of a and b.

$a \vee b$ Maximum of a and b.

$\lfloor x \rfloor$ Greatest integer not exceeding x.

Distributions

$\mathbf{Po}(\lambda)$ Poisson Distribution with mean $\lambda > 0$: if $X \stackrel{d}{=} \mathbf{Po}(\lambda)$, then $\mathbb{P}(X = j) = \dfrac{e^{-\lambda}\lambda^j}{j!}$, for all $j \in \mathbb{N}$.

$\mathbf{Po}(\lambda)\{A\}$ Cumulative Poisson probability on set $A \subset \mathbb{N}$: $\mathbf{Po}(\lambda)\{A\} = \displaystyle\sum_{j \in A} \frac{e^{-\lambda}\lambda^j}{j!}$.

$\mathbf{R}(\alpha, \beta)$ Uniform (or Rectangular) Distribution on the interval $[\alpha, \beta]$ $(\alpha < \beta)$: If $X \stackrel{d}{=} \mathbf{R}(\alpha, \beta)$, then $\mathbb{P}(X \leq x) = \dfrac{x - \alpha}{\beta - \alpha}$, for $x \in [\alpha, \beta]$.

Exp(λ) Exponential Distribution with mean λ^{-1}: If $X \stackrel{d}{=} \mathbf{Exp}(\lambda)$, then $\mathbb{P}(X \leq x) = 1 - e^{-\lambda x}$.

TruncExp(α, β, λ) Truncated Exponential Distribution on the interval $[\alpha, \beta]$ $(\alpha < \beta)$, with mean $\frac{1}{\lambda} + \frac{\alpha e^{-\lambda\alpha} - \beta e^{-\lambda\beta}}{e^{-\lambda\alpha} - e^{-\lambda\beta}}$: If $X \stackrel{d}{=} \mathbf{TruncExp}(\alpha, \beta, \lambda)$, then $\mathbb{P}(X \leq x) = \frac{e^{-\lambda\alpha} - e^{-\lambda x}}{e^{-\lambda\alpha} - e^{-\lambda\beta}}$.

Functions

$I_n(x)$ Modified Bessel function of order n: $I_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(-ie^{-i\theta})^n e^{-x\sin\theta}d\theta$.

$\mathrm{Q}(\alpha, \beta)$ First Order Marcum Q-Function: $\mathrm{Q}(\alpha, \beta) = \int_{\beta}^{\infty} xe^{-\left(\frac{x^2+\alpha^2}{2}\right)} I_0(\alpha x)dx$.

$\rho(\sigma, \tau)$ Marcum Q-Function (Detection probability form): $\rho(\sigma, \tau) = e^{-\sigma}\int_{\tau}^{\infty} e^{-\nu} I_0(2\sqrt{\sigma\nu})d\nu$. These are related via $\rho(\sigma, \tau) = \mathrm{Q}(\sqrt{2\sigma}, \sqrt{2\tau})$.

Erfc(z) Complementary error function: $\mathbf{Erfc}(z) = \frac{2}{\sqrt{\pi}}\int_{z}^{\infty} e^{-t^2}dt$.

Estimators

$\widehat{\Xi}_1 = \frac{1}{H}\sum_{j=1}^{H}\mathbb{P}(X \leq Y_j)$: Discrete Estimator based on Poisson sampling distribution.

$\widehat{\Xi}_2 = \frac{1}{M}\sum_{j=1}^{M}\sum_{k=0}^{Z_j} g_X(k)W(Z_j)$: Discrete Importance Sampling Estimator, with tilted sampling distribution.

$\widehat{\Xi}_3 = e^{-(\sigma+\tau)}\frac{1}{N}\sum_{j=1}^{N} I_0(2\sqrt{T_j\sigma})$: Continuous Estimator, based upon original Marcum Q-Function, using a Truncated Exponential sampling distribution $T_j \stackrel{d}{=} \mathbf{TruncExp}(\tau, \infty, 1)$.

$\widehat{\Xi}_4 = \frac{1}{2}[1 - e^{-2\sigma}I_0(2\sigma)] + e^{-\sigma}\widehat{\mathcal{I}}$: Continuous Estimator, based upon Theorem 1, Part (iii), using a uniform sampling distribution $T_j \stackrel{d}{=} \mathbf{R}(\tau \wedge \sigma, \tau \vee \sigma)$, where
$\widehat{\mathcal{I}} := ((\tau \vee \sigma) - (\tau \wedge \sigma))\,(\mathbb{I}[\sigma > \tau] - \mathbb{I}[\sigma < \tau])\,\frac{1}{N}\sum_{j=1}^{N} e^{-T_j} I_0(2\sqrt{T_j\sigma})$.

$\widehat{\Xi}_5 = \frac{1}{2}[1 - e^{-2\tau}I_0(2\tau)] + e^{-\sigma-\tau}I_0(2\sqrt{\sigma\tau}) + e^{-\tau}\widehat{\mathcal{J}}$: Continuous Estimator, based upon Theorem 1, Part (iv), using $T_j \stackrel{d}{=} \mathbf{R}(\tau \wedge \sigma, \tau \vee \sigma)$ sampling distribution, where
$\widehat{\mathcal{J}} := ((\tau \vee \sigma) - (\tau \wedge \sigma))\,(\mathbb{I}[\sigma > \tau] - \mathbb{I}[\sigma < \tau])\,\frac{1}{N}\sum_{j=1}^{N} e^{-T_j} I_0(2\sqrt{T_j\tau})$.

$\widehat{\Xi}_6 = \dfrac{1}{2}[1 - e^{-2\sigma} I_0(2\sigma)] + e^{-\sigma}\widehat{\mathcal{K}}$: Continuous Estimator, based upon Theorem 1, Part (iii), using sampling distribution $T_j \stackrel{d}{=} \textbf{TruncExp}(\tau \wedge \sigma, \tau \vee \sigma, 1)$, where $\widehat{\mathcal{K}} := (e^{-(\tau\wedge\sigma)} - e^{-(\tau\vee\sigma)})\,(\mathbb{I}[\sigma > \tau] - \mathbb{I}[\sigma < \tau])\,\dfrac{1}{N}\sum_{j=1}^{N} I_0(2\sqrt{T_j\sigma})$.

$\widehat{\Xi}_7 = \dfrac{1}{2}[1 - e^{-2\tau} I_0(2\tau)] + e^{-\sigma-\tau} I_0(2\sqrt{\sigma\tau}) + e^{-\tau}\widehat{\mathcal{L}}$: Continuous Estimator, based upon Theorem 1, Part (iv), using sampling distribution $T_j \stackrel{d}{=} \textbf{TruncExp}(\tau \wedge \sigma, \tau \vee \sigma, 1)$, where $\widehat{\mathcal{L}} := (e^{-(\tau\wedge\sigma)} - e^{-(\tau\vee\sigma)})\,(\mathbb{I}[\sigma > \tau] - \mathbb{I}[\sigma < \tau])\,\dfrac{1}{N}\sum_{j=1}^{N} I_0(2\sqrt{T_j\tau})$.

1 The Marcum Q-Function

The Marcum Q-Function [Marcum 1950, Marcum 1960, Marcum and Swerling 1960] has had a long association with the study of target detection by pulsed radars. In radar signal processing, the Generalised Marcum Q-Function is the detection probability of a number of incoherently integrated received signals, in a Gaussian clutter and noise environment [Helstrom 1968, Nuttall 1975 and Shnidman 1989]. It is also an important function in the study of digital communications. In the latter, it occurs in performance analysis related to partially coherent, differentiably coherent and noncoherent communications [Simon 1998 and Simon and Alouini 2003]. The Marcum Q-Function is a definite integral defined on a semi-infinite domain, whose integrand involves a modified Bessel function, and consequently no closed analytic result is available. Consequently, much research has been devoted to finding good approximations for it. Techniques employed to this end include numerical integration schemes and approximations based on the modified Bessel function in the integrand. Recently, some new expressions for the Marcum Q-Function have been derived, linking it to probabilities associated with independent Poisson random variables [Weinberg 2005]. These new representations are in a form that is readily adapted to Monte Carlo integration. Thus the purpose of this work is to investigate the application of the stochastic representations in [Weinberg 2005] to the Monte Carlo estimation of the Marcum Q-Function. In particular, we will be restricting attention to what is known as the standard Marcum Q-Function. The generalised Marcum Q-Function is considered in detail in [Weinberg 2005].

1.1 The Standard Marcum Q-Function

The first order, or standard, Marcum Q-Function is defined by the integral

$$\mathrm{Q}(\alpha,\beta) := \int_{\beta}^{\infty} x e^{-\left(\frac{x^2+\alpha^2}{2}\right)} I_0(\alpha x)dx, \tag{1}$$

where $I_0(\cdot)$ is the modified Bessel function, of the first kind, of order zero [Bowman 1958 and Tsypkin and Tsypkin 1988]. The integrand in (1) is the probability density function of a Rician distribution [Levanon 1988]. As pointed out in [Sarkies 1976], the latter is the distribution for the output of a linear law envelope detector with input signal of amplitude α and narrow band additive Gaussian noise with variance 1.

An equivalent form, which is slightly more natural for radar detection theory, can be obtained by letting $\alpha = \sqrt{2\sigma}$ and $\beta = \sqrt{2\tau}$. Under this transformation, we define the alternative form of (1):

$$\rho(\sigma,\tau) := e^{-\sigma} \int_{\tau}^{\infty} e^{-\nu} I_0(2\sqrt{\sigma\nu})d\nu. \tag{2}$$

In this form, σ is the constant received signal to noise ratio and τ is the normalised detection threshold [see Levanon 1988]. Throughout we will refer to (2) as the Marcum Q-Function, and restrict attention to this form, noting that results can easily be extended to (1) by using the fact that $\mathrm{Q}(\alpha,\beta) = \rho(\frac{\alpha^2}{2}, \frac{\beta^2}{2})$.

1.2 Estimating the Marcum $\mathbb{Q}$-Function

In view of the integrals (1) and (2), it is necessary to find good approximations for the Marcum Q-Function. There have been a number of schemes investigated over the years. These include applying numerical integration directly to (1) and (2). Two examples of such techniques are the application of Gauss-Laguerre integration in [Sarkies 1976], and saddlepoint integration in [Helstrom 1992]. These schemes generated good numerical approximations. Another class of techniques are those which utilise truncated Taylor series approximations applied to the Bessel function in (1) and (2). Such schemes are often referred to as recursive methods, and an excellent survey of such techniques can be found in [Shnidman 1989]. A major problem with recursive schemes applied to the estimation of the Marcum Q-Function is that they are prone to computational complexities. As pointed out in [Helstrom 1992], even for small parameter values in the Marcum Q-Function, a computer has to deal with underflow and overflow. In the case of large parameters, there will be a very large number of summations required, resulting in major round-off errors. From a practical point of view, there are merits and tradeoffs with both such schemes.

A class of numerical methods that has not been applied to the estimation of the Marcum Q-Function is Monte Carlo Methods [Ross 2002 and Srinivasan 2000]. This is likely to have been due to the fact that the expressions (1) and (2) do not appear to be in a useful form for such methods. On inspection of (2), for example, the only obvious choice of a Monte Carlo estimator is to use a Truncated Exponential sampling distribution. The only immediate alternative is to use Importance Sampling and sample from another distribution, and modify the integral using a weight function [Srinivasan 2000]. However, there is no obvious choice for such an Importance Sampling distribution.

In [Weinberg 2005] a number of new results were derived, linking (2) to a probability comparing a pair of independent Poisson random variables. This results in a very simple discrete Monte Carlo estimator of (2). Additionally, a stochastic representation of (2) is also derived in [Weinberg 2005], which leads to a number of possible continuous sampling distributions for Monte Carlo estimators.

1.3 Monte Carlo Methods

Monte Carlo Methods [Billingsley 1995, Robert and Casella 2004, Ross 2002, Srinivasan 2002] use statistical sampling techniques to estimate expectations of random variables. Consequently, this scheme can be used to approximate probabilities and integrals [Weinberg 2004 and Weinberg and Kyprianou 2005].

The *Strong Law of Large Numbers* (SLLN) [Billingsley 1995, Robert and Casella 2004, Ross 2002 and Srinivasan 2002] is the basis of the Monte Carlo approach to the estimation of statistical expectations. Suppose that $K \in \mathbb{N} - \{0\}$ and that we have a sequence $Z, Z_1, Z_2, \ldots, Z_K, \ldots$, consisting of independent and identically distributed random variables with mean $\mathbb{E}[Z]$. Then the simplest form of the SLLN states that

$$\lim_{K\to\infty} \frac{\sum_{j=1}^{K} Z_j}{K} = \mathbb{E}[Z], \tag{3}$$

except on a set of probability zero. Hence, the mean of a finite number of the random variables gives an approximation to the expectation $\mathbb{E}[Z]$. As K increases without bound, the approximation becomes more accurate. Thus, in order to estimate the mean $\mathbb{E}[Z]$, we generate a series of independent realisations of Z, and average them. The generation of realisations of random variables, both continuous and discrete, is described in detail in [Ross 2002].

We can apply (3) to a function of the sequence of original random variables. Specifically, if h is an integrable function, whose domain is the sample space of these random variables, then (3) implies that

$$\lim_{K\to\infty} \frac{\sum_{j=1}^{K} h(Z_j)}{K} = \mathbb{E}[h(Z)]. \tag{4}$$

Consequently the sum $\sum_{j=1}^{K} \frac{h(Z_j)}{K}$ in (4) can be used to approximate the expectation $\mathbb{E}[h(Z)]$. The approximations induced by (3) and (4) utilise statistical sampling to estimate an expectation, and thus have been referred to as *Monte Carlo Methods* [Robert and Casella 2004, Ross 2002 and Srinivasan 2002]. Although the SLLN guarantees the convergence of the sample mean in (3), there are a number of issues with estimators based upon this principle. The main difficulty is that the sample size K in (3) may have to be extremely large in order to achieve a prescribed variance. Sometimes it is possible to reduce the required sample size K by sampling from a different distribution, and modifying the underlying estimator to make it unbiased. Such techniques, often referred to as variance reduction techniques, are known as *Importance Sampling* [Srinivasan 2002].

1.4 Contributions of this Report

This report introduces the idea of applying Monte Carlo simulation schemes to the evaluation of (2). In particular, we introduce two Monte Carlo estimators of the Marcum Q-Function based upon discrete sampling distributions. One of these is based upon a Poisson association derived in [Weinberg 2005], while the second is an Importance Sampling estimator. Additionally, we investigate five Monte Carlo estimators, which use continuous sampling distributions. The first of these is based on direct sampling applied to (2), and uses a Truncated Exponential distribution referred to previously. The remaining four are based upon stochastic representations of the Marcum Q-Function. Two are the result of the expressions in [Weinberg 2005], while the second pair arise from a new stochastic form of (2), derived in this report.

The seven Monte Carlo estimators are compared to results derived from adaptive Simpson quadrature [Lyness and Kaganove 1976]. We also compare some of the estimators to results based upon partial sum series approximations of Taylor series representations of the Marcum Q-Function [Shnidman 1989]. We also examine the simulation gain of pairs of estimators, in an attempt to identify an optimal Monte Carlo estimator of (2).

2 Representations of the Marcum Q-Function

The key to Monte Carlo estimation of the Marcum Q-Function is to express it in a form that suggests a suitable sampling distribution. To this end, we present a number of results, derived in [Weinberg 2005], which readily suggest suitable sampling distributions. In addition, a new representation of the Marcum Q-Function is derived, which also suggests a number of possible Monte Carlo estimators. These expressions will be referred to as *probabilistic-based representations* of the Marcum Q-Function.

2.1 A General Result: Theorem 1

In [Weinberg 2005] a number of probabilistic or stochastic representations of the Marcum Q-Function are derived. These express (2) in terms of functions of probabilities of random variables. The following Theorem states these results, together with an entirely new result:

Theorem 1 *Suppose $\mathcal{P} = \{\mathcal{X}(\nu), \nu \in \mathbb{R}^+\}$ is a series of independent Poisson random variables with mean ν. Then the following are equivalent:*

(i) $\rho(\sigma, \tau)$ *is the Marcum* Q*-Function (2);*

(ii) $\rho(\sigma, \tau) = \mathbb{P}[X(\tau) \leq X(\sigma)]$*;*

(iii) $\rho(\sigma, \tau) = \frac{1}{2}[1 - e^{-2\sigma} I_0(2\sigma)] + e^{-\sigma} \int_\tau^\sigma e^{-\nu} I_0(2\sqrt{\nu\sigma}) d\nu$*;*

(iv) $\rho(\sigma, \tau) = \frac{1}{2}[1 - e^{-2\tau} I_0(2\tau)] + e^{-\sigma-\tau} I_0(2\sqrt{\sigma\tau}) + e^{-\tau} \int_\tau^\sigma e^{-\nu} I_0(2\sqrt{\nu\tau}) d\nu$*.*

The proof that (ii) and (iii) are eqivalent to the Marcum Q-Function can be found in [Weinberg 2005]. Expression (ii) shows that (2) is the same as a comparison of two Poisson random variables, one with mean being the signal to noise ratio σ, while the second has as mean the threshold τ. This gives a very intuitive interpretation to the Marcum Q-Function, which can be found in [Weinberg 2005]. Result (iv) is an entirely new representation of (2), and was derived using the symmetry relationship of the Marcum Q-Function [see Schwartz, Bennett and Stein 1996].

To prove Theorem 1, we need only derive (iv). We require the two following technical Lemmas:

Lemma 1 *For the Marcum* Q*-Function* $\rho(\cdot, \cdot)$*,*

$$\rho(\sigma, \sigma) = \frac{1}{2}[1 - e^{-2\sigma} I_0(2\sigma)]. \tag{5}$$

The proof of Lemma 1 can be found in Appendix A of [Weinberg 2005]. Note $\rho(\sigma,\sigma)$ is the detection probability corresponding to the case when the threshold and signal to noise ratio are equal.

The next Lemma is the well-known symmetry relation of the Marcum Q-Function:

Lemma 2 *The Marcum* Q*-Function* $\rho(\cdot,\cdot)$ *has the property that*

$$\rho(\sigma,\tau)+\rho(\tau,\sigma)=1+e^{-\sigma-\tau}I_0(2\sqrt{\sigma\tau}). \tag{6}$$

Proof : Although this is a well-known result, and can be found in [Schwartz, Bennett and Stein 1996], we present a new probabilistic proof. Assume that $X_1,X_2,X_3,X_4\in\mathcal{P}$ such that $X_1\stackrel{d}{=}X_4$ and $X_2\stackrel{d}{=}X_3$. Then Theorem 1 Part (ii) implies

$$\rho(\tau,\sigma)=\mathbb{P}[X_1(\sigma)\le X_2(\tau)] \tag{7}$$

and

$$\rho(\sigma,\tau)=\mathbb{P}[X_3(\tau)\le X_4(\sigma)]. \tag{8}$$

Hence it follows that

$$\begin{aligned}\rho(\sigma,\tau)+\rho(\tau,\sigma) &= \mathbb{P}[X_3(\tau)=X_4(\sigma)]+\mathbb{P}[X_3(\tau)<X_4(\sigma)]\\ &\quad +\mathbb{P}[X_1(\sigma)=X_2(\tau)]+\mathbb{P}[X_1(\sigma)<X_2(\tau)]. \end{aligned}\tag{9}$$

By construction it follows that

$$\mathbb{P}[X_1(\sigma)=X_2(\tau)]=\mathbb{P}[X_3(\tau)=X_4(\sigma)]. \tag{10}$$

Thus, by applying (10) to (9), we deduce

$$\begin{aligned}\rho(\sigma,\tau)+\rho(\tau,\sigma) &= 2\mathbb{P}[X(\sigma)=X(\tau)]+\mathbb{P}[X(\tau)<X(\sigma)]\\ &\quad +\mathbb{P}[X(\sigma)<X(\tau)]\\ &= 2\mathbb{P}[X(\sigma)=X(\tau)]+\mathbb{P}[X(\tau)\neq X(\sigma)]\\ &= 1+\mathbb{P}[X(\sigma)=X(\tau)]. \end{aligned}\tag{11}$$

The difference of two independent Poisson distributions is known as a Skellam distribution [Skellam 1946], and it can be shown that the zero probability of such a distribution implies that

$$\mathbb{P}[X(\sigma)=X(\tau)]=\sum_{k=0}^{\infty}\frac{e^{-\sigma-\tau}(\sigma\tau)^k}{k!^2}. \tag{12}$$

Also, the modified Bessel function of order zero has Taylor series expansion

$$I_0(z) = \sum_{k=0}^{\infty} \frac{(\frac{z^2}{4})^k}{k!^2} \tag{13}$$

[Bowman 1958].

Hence, with the choice of $z = 2\sqrt{\sigma\tau}$, we have

$$\mathbb{P}[X(\sigma) = X(\tau)] = e^{-\sigma-\tau} I_0(2\sqrt{\sigma\tau}). \tag{14}$$

Consequently, by an application of (14) to (11), we deduce that

$$\rho(\sigma, \tau) + \rho(\tau, \sigma) = 1 + e^{-\sigma-\tau} I_0(2\sqrt{\sigma\tau}), \tag{15}$$

which completes the proof of the Lemma.

□

We are now in a position to prove Part (iv) of Theorem 1.

Proof of Theorem 1, Part (iv):

By interchanging σ and τ in Part (iii) in Theorem 1, we note that

$$\rho(\tau, \sigma) = \rho(\tau, \tau) + \int_{\sigma}^{\tau} \mathbb{P}[X(\nu) = X(\tau)] d\nu. \tag{16}$$

An application of (16) to the symmetry relation in Lemma 2, and applying Lemma 1, we deduce that

$$\begin{aligned}
\rho(\sigma, \tau) &= 1 + e^{-\sigma-\tau} I_0(2\sqrt{\sigma\tau}) - \rho(\tau, \tau) - \int_{\sigma}^{\tau} \mathbb{P}[X(\nu) = X(\tau)] d\nu \\
&= 1 + e^{-\sigma-\tau} I_0(2\sqrt{\sigma\tau}) - \frac{1}{2}[1 + e^{-2\tau} I_0(2\tau)] \\
&\quad + \int_{\tau}^{\sigma} \mathbb{P}[X(\nu) = X(\tau)] d\nu \\
&= \frac{1}{2} + e^{-\sigma-\tau} I_0(2\sqrt{\sigma\tau}) - \frac{1}{2}[1 + e^{-2\tau} I_0(2\tau)] \\
&\quad + \int_{\tau}^{\sigma} \mathbb{P}[X(\nu) = X(\tau)] d\nu \\
&= \frac{1}{2}[1 - e^{-2\tau} I_0(2\tau)] + e^{-\sigma-\tau} I_0(2\sqrt{\sigma\tau}) + \int_{\tau}^{\sigma} \mathbb{P}[X(\nu) = X(\tau)] d\nu.
\end{aligned} \tag{17}$$

The proof is completed by recalling that $\mathbb{P}[X(\nu) = X(\tau)] = e^{-\nu-\tau} I_0(2\sqrt{\nu\tau})$, and applying this to (17).

$\square$

In the next Section we derive a number of estimators, based upon the results of Theorem 1.

3 Monte Carlo Estimators of the Marcum $\mathbb{Q}$-Function

We are now in a position to introduce a series of Monte Carlo sampling estimators of the Marcum Q-Function (2). Estimators based upon sampling from both discrete and continuous distributions will be considered. At this stage we limit our attention to introducing these estimators. For reference, Appendix A contains some details on the calculation of variances of random variables. Additionally, Appendix B outlines how realisations of random variables, from a prescribed distribution, can be obtained.

3.1 Discrete Estimators

To begin, we consider a number of estimators of the Marcum Q-Function using discrete sampling distributions. Firstly, we illustrate how the Monte Carlo scheme works in this case. With reference to (4), we suppose Z is a discrete random variable with support $\mathbb{N} = \{0, 1, 2, \ldots\}$, and h is a function with the same support. We want to estimate the expectation $\mathbb{E}[h(Z)]$. A basic Monte Carlo estimator of this expectation can be based on

$$\mathbb{E}[h(Z)] = \sum_{k=0}^{\infty} h(k)\mathbb{P}[Z = k] \approx \frac{1}{K}\sum_{j=1}^{K} h(Z_j), \tag{18}$$

where the sequence $Z_1, Z_2, \ldots, Z_K$ consists of independent and identically distributed copies of the random variable Z. Throughout we will employ the statistical convention of denoting an estimator by using a hat over its symbol. Hence, we write $\widehat{\aleph}$ to represent the estimator in (18), so that

$$\widehat{\aleph} = \frac{1}{K}\sum_{j=1}^{K} h(Z_j). \tag{19}$$

3.1.1 A Standard Monte Carlo Estimator

The first Monte Carlo estimator we consider is based directly on Theorem 1, Part (ii). This result shows that the Marcum Q-Function (2) can be represented as a probability of the form $\mathbb{P}(X \leq Y)$, where X and Y are independent (Poisson) random variables with support $\mathbb{N}$. With reference to (18), we choose a two-dimensional version of h: $h(x, y) = \mathbb{I}[x \leq y]$, where $\mathbb{I}$ is the indicator function. This means that $h(x, y) = 1$ if $x \leq y$ and is zero otherwise. We also let $\psi(X, Y) = \mathbb{E}h(X, Y) \equiv \mathbb{P}[X \leq Y]$, which is the probability under investigation.

Then the standard Monte Carlo estimator of $\psi(X, Y)$ is

$$\widehat{\Xi} \;=\; \frac{1}{K}\sum_{j=1}^{K} h(X_j, Y_j)$$

$$= \frac{1}{K}\sum_{j=1}^{K}\sum_{k=0}^{Y_j} \mathbb{I}[X_j = k], \tag{20}$$

where the pairs (X_j, Y_j) consist of independent and identically distributed copies of (X, Y). The generation of realisations of Poisson random variables is described in [Ross 2002], and also in Appendix B, to which the reader is referred.

It is not difficult to show that (20) is an unbiased estimator of $\psi(X, Y)$, meaning that $\mathbb{E}[\widehat{\Xi}] = \psi(X, Y)$, so that the estimator is centred on the probability it is estimating. Its variance can be shown to be

$$\mathbb{V}[\widehat{\Xi}] = \frac{1}{K}\left[\psi(X,Y) - \psi(X,Y)^2\right]. \tag{21}$$

The expression in (21) shows that as the sample size increases without bound, the estimator's variation from its expected value decreases to zero. The issue of interest is how large must N be so that this variance is within a prescribed tolerance. Suppose we require $\mathbb{V}(\widehat{P}) \leq \epsilon$, for some $\epsilon > 0$. Using (21), it is not difficult to see that we need to choose

$$K = \left\lfloor \frac{\psi(X,Y) - \psi(X,Y)^2}{\epsilon} \right\rfloor + 1, \tag{22}$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding x. Thus, K is of order $\frac{1}{\epsilon}$, unless the probability $\psi(X, Y)$ is very small relative to ϵ. Equation (22) shows the inherent problems one faces with Monte Carlo estimation. The SLLN guarantees that the estimator will converge, but the tradeoff is that this might be at the expense of a very large number of simulation runs. There is an exception to this. In view of the variance (21), if the probability $\psi(X, Y)$ is very small, then the variance will also be very small, independently of the number of simulation runs K. The probability $\psi(X, Y)$ will be very small when the random variable X is significantly larger than Y. This case implies Monte Carlo methods will have the best performance for the estimation of probabilities of rare events. Nevertheless, we will show that an alternative to (20) can be produced, which is a globally more efficient estimator.

3.1.2 A Poisson-Based Sampling Estimator: $\widehat{\Xi}_1$

The following approach is motivated by the work of [Srinivasan 2000] on the so-called G-function estimator, and also by the analysis of [Bucklew 2003] on bias point selection.

Note that, since we are assuming X and Y are independent, we can write

$$\mathbb{P}[X \leq Y] = \sum_{k=0}^{\infty} \mathbb{P}[X \leq k]\mathbb{P}[Y = k], \tag{23}$$

and so, in view of (18), a Monte Carlo estimator of (23) is

$$\widehat{\Xi}_1 = \frac{1}{H}\sum_{j=1}^{H} \mathbb{P}[X \leq Y_j]$$

$$= \frac{1}{H}\sum_{j=1}^{H}\sum_{k=0}^{Y_j} g_X(k), \tag{24}$$

where each Y_j is an independent realisation of Y, and we define $g_X(k) = \mathbb{P}[X = k]$. Hence, we can estimate probabilities of the form $\mathbb{P}[X \leq Y]$ by generating independent realisations of Y and averaging the cumulative distribution function over these values. Expression (23) provides a means of compression of the probability of interest, analogous to that used in [Srinivasan 2000]. Sampling from a Poisson distribution, as remarked previously, can be easily achieved through any of the algorithms given in [Ross 2002] and Appendix B.

We now examine the estimator (24) more closely. Firstly, it is not difficult to show that it is also an unbiased estimator of $\psi(X,Y)$. To see this, observe that

$$\begin{aligned}
\mathbb{E}[\widehat{\Xi}_1] &= \sum_{m=0}^{\infty}\mathbb{P}[Y=m]\sum_{k=0}^{m} g_X(k) \\
&= \sum_{m=0}^{\infty}\mathbb{P}[Y=m]\mathbb{P}[X\leq m] \\
&= \mathbb{P}[X \leq Y].
\end{aligned}$$

Secondly, it is not difficult to show its variance is given by

$$\mathbb{V}[\widehat{\Xi}_1] = \frac{1}{H}\left[\mathbb{E}\left[\sum_{k=0}^{Y} g_X(k)\right]^2 - \psi(X,Y)^2\right]. \tag{25}$$

Observe that the sum in the first expectation in (25) is a (random) sum of probabilities of the same random variable X, and so is bounded by one. This implies that (25) is smaller than (21), for the same number of simulations ($K = H$), and consequently estimator (24) is more efficient than (20). Hence we will not consider the standard Monte Carlo estimator (20) any further.

The simulation gain, of a pair of estimators, is a quantitative measure of the improvement one Monte Carlo estimator has over another, in terms of reducing the number of simulations. For the same level of variance, we are interested in the size of the ratio of K and H. By equating the expressions (21) and (25), we obtain

$$\Gamma = \frac{K}{H} = \frac{\psi(X,Y) - \psi(X,Y)^2}{\mathbb{E}\left[\sum_{k=0}^{Y} g_X(k)\right]^2 - \psi(X,Y)^2}, \tag{26}$$

and the previous remarks imply that $\Gamma > 1$. Consequently, for the same level of variance, the estimator (24) requires less simulation runs than (20). To determine the exact level of improvement is a somewhat complicated exercise. This is due to the fact that both the variances (21) and (25) depend on the unknown probability $\psi(X,Y)$, as does the gain (26). Secondly, the variance (25), and so (26), both depend on the expectation

$\mathbb{E}\left[\sum_{k=0}^{Y} g_X(k)\right]^2$, which is not readily evaluated. Both these difficulties can be partially resolved using estimation. This will at least give a partial understanding of the potential improvement provided by the estimator (24).

3.1.3 An Importance Sampling Estimator: $\widehat{\Xi}_2$

It is now worth considering whether an Importance Sampling estimator can provide an improvement on the estimator (24). Importance Sampling (IS) [Robert and Casella 2004, Ross 2002 and Srinivasan 2002] has been developed in an attempt to address the sample size issues associated with Monte Carlo methods. This is a variance reduction technique, which attempts to reduce the Monte Carlo estimator's variance by sampling from a distribution not directly suggested by the probability being estimated. In the current context, one would introduce biasing distributions, which would be used in (20) instead of X_j and Y_j. In order to make the resulting estimator unbiased, it is weighted at each point by a weight function. As pointed out in [Srinivasan 2002], these biasing distributions are chosen in an attempt to increase the distribution of points relevant to the estimation, or in other words, sample points that are important to the Monte Carlo simulation. A consequence of the successful achievement of this is that the resulting estimator's variance should be reduced. This will also result in a reduction in simulation runs, when compared to a standard Monte Carlo estimator.

Much work has been devoted to the design of efficient IS biasing distributions [Srinivasan 2002]. However, it is important to remember that Monte Carlo IS techniques tend to work best when estimating probabilities associated with rare events, such as false alarm probabilities in CFAR processes [see Ross 2002 and Srinivasan 2002]. In the current context, we are interested in the Marcum Q-Function (2), which take values in a full spectrum of possibilities. Hence it is possible that Importance Sampling will not improve significantly the performance of Monte Carlo estimators of the Marcum Q-Function.

We attempt the construction of an Importance Sampling estimator based on (24). The key to this is to replace the random variables $Y_j, j \in \{1, 2, \ldots, H\}$ with a new biasing distribution Z_j, for j in the same indexing set, and weighting the estimator (24) at each point, to make the resulting estimator unbiased. Such an estimator can be defined as

$$\widehat{\Xi}_2 = \frac{1}{M}\sum_{j=1}^{M}\sum_{k=0}^{Z_j} g_X(k) W(Z_j), \tag{27}$$

where the random variables Z_j are independent and identically distributed copies of the biasing random variable Z. The function $W(\cdot)$ in (27) is a weight function, which is chosen to make the estimator unbiased for $\psi(X, Y)$. It can be shown that the latter necessitates the choice of

$$W(k) = \frac{\mathbb{P}[Y = k]}{\mathbb{P}[Z = k]}, \tag{28}$$

which also shows that we must ensure that any choice made for the biasing distribution does not have zero probabilities on its support. This automatically excludes the choice of a truncated Poisson distribution, which would have been a somewhat natural choice. The latter is the case because a Poisson distribution is centered on its mean, and its variance is also equal to its mean. Thus a truncated Poisson could be constructed that gives more likelihood near its mean value.

The variance of (27) can be shown to be

$$\begin{aligned}\mathbb{V}[\widehat{\Xi}_2] &= \frac{1}{M}\left[\mathbb{E}\left[\sum_{k=0}^{Z} g_X(k)W(Z)\right]^2 - \psi(X,Y)^2\right] \\ &= \frac{1}{M}\left[\mathbb{E}\left[\left(\sum_{k=0}^{Y} g_X(k)\right)^2 W(Y)\right] - \psi(X,Y)^2\right],\end{aligned} \tag{29}$$

where the latter equality follows by applying the definition of the weight function (28), and expanding out the expectation. Comparing (29) to (25), we see that if a biasing distribution can be chosen so that the weight function never exceeds unity, the corresponding Importance Sampling estimator will be more efficient. In the context of interest, since the biasing distribution will have the same support as Y, namely the nonnegative integers, this property will not hold [see Srinivasan, 2002]. There are a number of Importance Sampling biasing distributions that have been studied in the literature. These have been developed by using properties of the unique optimal biasing distribution associated with Importance Sampling techniques [Srinivasan, 2002]. To illustrate this in our current situation, consider the choice of biasing distribution with point probabilities

$$\mathbb{P}[Z=m] = \psi(X,Y)^{-1}\mathbb{P}[Y=m]\sum_{k=0}^{m} g_X(k). \tag{30}$$

Applying (30) to the variance (29), we see that the corresponding estimator (27) has zero variance. The distribution (30) cannot be used in practice, because it depends on the unknown probability of interest, namely $\psi(X,Y)$. However, as pointed out in [Srinivasan 2002], its form suggests how potential biasing distributions can be constructed. Specifically, it suggests a biasing distribution should be proportional to the original distribution, and concentrated on the event or region of interest. Based on such observations, potential biasing distributions include scaling and translation applied to the original distribution [Srinivasan 2002], exponential twisting or tilting [Ross 2002 and Srinivasan 2002] and Chernoff Importance Sampling distributions [Gerlach 1999].

We consider the case of a discrete tilted biasing distribution [Ross 2002]. Such a distribution has point probabilities given by

$$\mathbb{P}[Z=k] = \mathbb{P}[Z=k|\theta] = \frac{\theta^k\mathbb{P}[Y=k]}{\sum_{m=0}^{\infty}\theta^m\mathbb{P}[Y=m]}, \tag{31}$$

for all $k \in \mathbb{N}$, where $\theta > 0$ is a biasing parameter. Observe that the normalising constant on the denominator of (31) is the probability generating function of Y [see Billingsley 1995 and Durrett 1996]. We assume that Y is Poisson with parameter λ. Consequently, it can be shown that $\sum_{m=0}^{\infty} \theta^m \mathbb{P}[Y = m] = e^{-\lambda(1-\theta)}$, and hence (31) becomes

$$\mathbb{P}[Z = k] = \frac{e^{-\lambda\theta}(\lambda\theta)^k}{k!}, \tag{32}$$

which implies that the biasing distribution is also Poisson, but with a parameter of $\lambda\theta$. Additionally, it follows that the weight function (28) is $W(k) = \theta^{-k} e^{-\lambda(1-\theta)}$. This weight function implies the variance (29) becomes

$$\mathbb{V}[\widehat{\Xi}_2] = \frac{1}{M}\left[e^{-\lambda(1-\theta)} \mathbb{E}\left[\theta^{-Y} \left(\sum_{k=0}^{Y} g_X(k) \right)^2 \right] - \psi(X,Y)^2 \right]. \tag{33}$$

An issue with the variance (33) is that if $\theta < 1$, the term θ^{-Y} in the expectation component of (33) will have the potential to grow exponentially. This is due to the fact that Y takes values in the nonnegative integers. Also, with the choice of $\theta > 1$, the term $e^{-\lambda(1-\theta)}$ will also grow exponentially, but not in such a dynamic way. In this case, the term θ^{-Y} will cause the expectation in (33) to decrease exponentially, and has the potential to control the behaviour of the multiplier term. Hence we restrict attention to the case where $\theta \geq 1$. Our interest is whether a $\theta > 1$ can be found, such that the variance (33) is smaller than (25), when $H = M$. As before, we let the simulation gain be $\Gamma = \frac{H}{M}$. Then for the same variance in (25) and (33),

$$\Gamma = \frac{\mathbb{E}\left[\sum_{k=0}^{Y} g_X(k) \right]^2 - \psi(X,Y)^2}{\left[e^{-\lambda(1-\theta)} \mathbb{E}\left[\theta^{-Y} \left(\sum_{k=0}^{Y} g_X(k) \right)^2 \right] - \psi(X,Y)^2 \right]}. \tag{34}$$

In contrast to the gain (26), it is not mathematically straightforward to determine whether the gain (34) exceeds 1, for particular choices of θ. For specific choices of the free parameters one can investigate this gain numerically. Also, it is possible to attempt to choose a θ that minimises the variance (33), by employing a stochastic Newton recursion, as in [Srinivasan 2000]. The disadvantage of the latter is that it necessitates the introduction of two additional Monte Carlo estimators, as well as a recursion scheme, which can add considerably to the numerical computation times. We will examine these gains further in Section 4.

3.2 Continuous Estimators

Estimators based upon continuous sampling distributions are now considered. On inspection of the Marcum Q-Function integral (2), continuous sampling distributions are the most obvious approach. In such cases, we are again interested in estimating the expectation $\mathbb{E}[h(Z)]$, but we assume that Z has a density g on a subset of the real line, $\Omega \subset \mathbb{R}$. Then, in view of (4), this implies

$$\mathbb{E}[h(Z)] = \int_{\Omega} h(z)g(z)dz \approx \frac{1}{K}\sum_{j=1}^{K} h(Z_j), \tag{35}$$

where each Z_j is generated from a random variable with density g. The Marcum Q-Function integral (2) has an exponential term in its integrand, which can be weighted to produce a density that can then be used to construct a sampling distribution. We will consider this estimator, as well as a number of others that can be derived from Theorem 1.

3.2.1 Estimator Based on Original Marcum Q-Function Integral: $\widehat{\Xi}_3$

As remarked previously, an obvious choice for biasing distribution of (2) is a Truncated Exponential distribution, with this distribution the restriction of the standard exponential distribution to the interval $[\tau, \infty)$. We denote this distribution by $\mathbf{TruncExp}(\tau, \infty, 1)$, and its corresponding density is $f_T(\nu) = e^{\tau-\nu}$, for $\nu \geq \tau$. By scaling the Marcum Q-Function integral (2) by $e^{-\tau}$, we arrive at the estimator

$$\widehat{\Xi}_3 = e^{-(\sigma+\tau)} \frac{1}{N} \sum_{j=1}^{N} I_0(2\sqrt{T_j \sigma}), \tag{36}$$

where each T_j is generated by independently sampling from the $\mathbf{TruncExp}(\tau, \infty, 1)$ distribution. Sampling from the latter is relatively straightforward, since it only requires one to sample from a uniform distribution on the unit interval [0,1], and then apply a simple transformation. Specifically, since the cumulative distribution function of $T \stackrel{d}{=} \mathbf{TruncExp}(\tau, \infty, 1)$ is $F_T(\nu) = 1 - e^{\tau-\nu}$, for $\nu \geq \tau$, and its inverse is $F_T^{-1}(\nu) = \tau - \log(1-\nu)$, it follows from Appendix B that T can be simulated using $\tau - \log(R)$, where $R \stackrel{d}{=} \mathbf{R}[0, 1]$.

It is not difficult to show this is also an unbiased estimator of (2). Observe that

$$\begin{aligned} \mathbb{E}[\widehat{\Xi}_3] &= e^{-(\sigma+\tau)} \mathbb{E}[I_0(2\sqrt{T\sigma})] \\ &= e^{-(\sigma+\tau)} \int_\tau^\infty e^{\tau-\nu} I_0(2\sqrt{\nu\sigma}) d\nu \\ &= \int_\tau^\infty e^{-\sigma-\nu} I_0(2\sqrt{\nu\sigma}) d\nu, \end{aligned}$$

which is (2), implying $\widehat{\Xi}_3$ is unbiased.

The variance of estimator $\widehat{\Xi}_3$ is given by the expression

$$\begin{aligned} \mathbb{V}[\widehat{\Xi}_3] &= e^{-2(\sigma+\tau)} \frac{1}{N} \mathbb{V}[I_0(2\sqrt{T\sigma})] \\ &= e^{-2(\sigma+\tau)} \frac{1}{N} \left[\mathbb{E}[I_0^2(2\sqrt{T\sigma})] - \left(\mathbb{E}[I_0(2\sqrt{T\sigma})] \right)^2 \right]. \end{aligned} \tag{37}$$

Using the definition of T, it follows that

$$\mathbb{E}[I_0^2(2\sqrt{T\sigma})] = \int_\tau^\infty e^{\tau-\nu} I_0^2(2\sqrt{\nu\sigma}) d\nu, \tag{38}$$

and also

$$\mathbb{E}[I_0(2\sqrt{T\sigma})] = \int_\tau^\infty e^{\tau-\nu} I_0(2\sqrt{\nu\sigma})d\nu, \tag{39}$$

so that numerical integration techniques can be applied to both (38) and (39), which then yield a numerical estimate of (37). We will consider continuous estimator's variances in more detail in Section 4.

3.2.2 An Estimator Based on Theorem 1 Part (iii), with Uniform Sampling Distribution: $\widehat{\Xi}_4$

We now consider estimators of the Marcum Q-Function, based upon the results of Theorem 1, Parts (iii) and (iv), that use continuous sampling distributions. The first of these is based upon a uniform samping distribution applied to Part (iii) of Theorem 1.

Let T be a uniformly distributed random variable on the interval $[\tau \wedge \sigma, \tau \vee \sigma]$, so that $T \stackrel{d}{=} \mathbf{R}(\tau \wedge \sigma, \tau \vee \sigma)$. Such a random variable has density $f_T(\nu) = \frac{1}{(\tau\wedge\sigma)-(\tau\vee\sigma)}$, for $\tau \wedge \sigma < \nu < \tau \vee \sigma$. Since this density is independent of its free variable ν, we can insert it into the expression in Part (iii) of Theorem 1, and multiply the integral by its reciprocal to balance the equation. In view of this, we focus on the integral component of Part (iii) in Theorem 1.

Observe that

$$\begin{aligned} \mathcal{I} &:= \int_\tau^\sigma e^{-\nu} I_0(2\sqrt{\nu\sigma})d\nu \qquad (40) \\ &\equiv \int_{\tau\wedge\sigma}^{\tau\vee\sigma} e^{-\nu} I_0(2\sqrt{\nu\sigma})d\nu \times (\mathbb{I}[\sigma > \tau] - \mathbb{I}[\sigma < \tau]) \\ &= [(\tau \vee \sigma) - (\tau \wedge \sigma)] \int_{\tau\wedge\sigma}^{\tau\vee\sigma} f_R(\nu) e^{-\nu} I_0(2\sqrt{\nu\sigma})d\nu \\ &\quad \times (\mathbb{I}[\sigma > \tau] - \mathbb{I}[\sigma < \tau]) \,. \qquad (41) \end{aligned}$$

Consequently, (41) is in a suitable form to apply the SLLN (4) to produce a Monte Carlo estimator. Specifically, if we let $T_j \stackrel{d}{=} \mathbf{R}(\tau \wedge \sigma, \tau \vee \sigma)$ be a series of independent and identically distributed uniform random variables, then we can derive estimates of the Marcum Q-Function from

$$\widehat{\Xi}_4 = \frac{1}{2}[1 - e^{-2\sigma} I_0(2\sigma)] + e^{-\sigma}\widehat{\mathcal{I}}, \tag{42}$$

where $\mathcal{I}$ is estimated from

$$\widehat{\mathcal{I}} = ((\tau \vee \sigma) - (\tau \wedge \sigma)) (\mathbb{I}[\sigma > \tau] - \mathbb{I}[\sigma < \tau]) \frac{1}{N} \sum_{j=1}^N e^{-T_j} I_0(2\sqrt{T_j \sigma}). \tag{43}$$

It is relatively straightforward to show that (42) is an unbiased estimator of the Marcum Q-Function $\rho(\sigma, \tau)$. Its variance, however, is more involved. Note that the deterministic components in (42) contribute nothing to the estimator's variance. Also, note that the square of the difference of the indicator functions in (43) will be unity. Hence it follows that

$$\begin{aligned}
\mathbb{W}[\widehat{\Xi}_4] &= e^{-2\sigma}\mathbb{W}[\widehat{\mathcal{I}}] \\
&= e^{-2\sigma}\frac{1}{N}\left[(\tau\vee\sigma)-(\tau\wedge\sigma)\right]^2\mathbb{W}\left[e^{-T}I_0(2\sqrt{T\sigma})\right] \\
&= \frac{e^{-2\sigma}}{N}\left[(\tau\vee\sigma)-(\tau\wedge\sigma)\right]^2 \\
&\quad\times\left(\mathbb{E}[e^{-2T}I_0^2(2\sqrt{T\sigma})]-\left(\mathbb{E}[e^{-T}I_0(2\sqrt{T\sigma})]\right)^2\right). \qquad (44)
\end{aligned}$$

We can, as previously, use the definition of T to write the expectations in (44) as integrals, but we do not include these here. The two expression (37) and (44), for the variances of estimators $\widehat{\Xi}_3$ and $\widehat{\Xi}_4$ respectively, do not provide much insight into the appropriate estimator's performance *per se*. Mathematically, it is quite difficult to work out closed form expressions that lead to useful simulation gain estimates. We will thus produce numerical estimates and plots of simulation gains in Section 4. These expressions for estimator's variance have been included for completeness.

3.2.3 An Estimator Based upon Theorem 1, Part (iv) using a Uniform Sampling Distribution: $\widehat{\Xi}_5$

This estimator also uses a uniform sampling distribution, but is instead based upon Part (iv) of Theorem 1. As previously, we let $T \stackrel{d}{=} \mathbf{R}(\tau\wedge\sigma,\tau\vee\sigma)$. The only part that involves Monte Carlo estimation is the integral component in Part (iv) of Theorem 1. As in the derivation of (41), observe that we can write

$$\mathcal{J} := \int_\tau^\sigma e^{-\nu}I_0(2\sqrt{\nu\tau})d\nu \qquad (45)$$

$$\equiv \int_{\tau\wedge\sigma}^{\tau\vee\sigma} e^{-\nu}I_0(2\sqrt{\nu\tau})d\nu\times(\mathbb{I}[\sigma>\tau]-\mathbb{I}[\sigma<\tau]). \qquad (46)$$

Thus, as in the argument to construct the estimator (42), we can use (46) to produce the estimator

$$\widehat{\Xi}_5 = \frac{1}{2}[1-e^{-2\tau}I_0(2\tau)]+e^{-\sigma-\tau}I_0(2\sqrt{\sigma\tau})+e^{-\tau}\widehat{\mathcal{J}}, \qquad (47)$$

where the integral (45) is estimated from

$$\widehat{\mathcal{J}} = ((\tau\vee\sigma)-(\tau\wedge\sigma))\left(\mathbb{I}[\sigma>\tau]-\mathbb{I}[\sigma<\tau]\right)\frac{1}{N}\sum_{j=1}^N e^{-T_j}I_0(2\sqrt{T_j\tau}), \qquad (48)$$

and each T_j is an independent realisation of T. Again, it is relatively straightforward to show that (47) is an unbiased estimator of the Marcum Q-Function. Also, it is not difficult to write down an expression for its variance. In particular, it can be shown that

$$\begin{aligned}\mathbb{V}[\widehat{\Xi}_5] &= \frac{e^{-2\tau}}{N}\left[(\tau \vee \sigma) - (\tau \wedge \sigma)\right]^2 \\ &\quad \times \left(\mathbb{E}[e^{-2T} I_0^2(2\sqrt{T\tau})] - \left(\mathbb{E}[e^{-T} I_0(2\sqrt{T\tau})]\right)^2\right). \end{aligned} \tag{49}$$

3.2.4 Estimator Based upon Theorem 1, Part (iii) with Truncated Exponential Sampling Distribution: $\widehat{\Xi}_6$

We now consider using a sampling distribution based upon a truncated exponential family. In this case, we consider Part (iii) of Theorem 1, and in view of the integral (40), we introduce a Truncated Exponential distribution $T \stackrel{d}{=} \mathbf{TruncExp}(\tau \wedge \sigma, \tau \vee \sigma, 1)$. Such a distribution has density $f_T(t) = \frac{e^{-t}}{e^{-(\tau\wedge\sigma)} - e^{-(\tau\vee\sigma)}}$, and can be simulated using $-\log[e^{-(\tau\wedge\sigma)} - R[e^{-(\tau\wedge\sigma)} - e^{-(\tau\vee\sigma)}]$, where $R \stackrel{d}{=} \mathbf{R}[0,1]$.

Let $\widehat{\mathcal{K}}$ be the estimator

$$\widehat{\mathcal{K}} = (e^{-(\tau\wedge\sigma)} - e^{-(\tau\vee\sigma)})\,(\mathbb{I}[\sigma > \tau] - \mathbb{I}[\sigma < \tau])\,\frac{1}{N}\sum_{j=1}^{N} I_0(2\sqrt{T_j\sigma}), \tag{50}$$

where each $T_j \stackrel{d}{=} \mathbf{TruncExp}(\tau \wedge \sigma, \tau \vee \sigma, 1)$. Then we can define the estimator

$$\widehat{\Xi}_6 = \frac{1}{2}[1 - e^{-2\sigma} I_0(2\sigma)] + e^{-\sigma}\widehat{\mathcal{K}}. \tag{51}$$

It is again not difficult to show this is an unbiased estimator of (2), with variance given by

$$\begin{aligned}\mathbb{V}[\widehat{\Xi}_6] &= \frac{e^{-2\sigma}}{N}\left[e^{-(\tau\wedge\sigma)} - e^{-(\tau\vee\sigma)}\right]^2 \\ &\quad \times \left(\mathbb{E}[I_0^2(2\sqrt{T\sigma})] - \left(\mathbb{E}[I_0(2\sqrt{T\sigma})]\right)^2\right). \end{aligned} \tag{52}$$

3.2.5 An Estimator Based upon Part (iv) of Theorem 1 with Truncated Exponential Sampling Distribution: $\widehat{\Xi}_7$

The final estimator we consider is also based upon a Truncated Exponential distribution, using Part (iv) of Theorem 1. Let $\widehat{\mathcal{L}}$ be the estimator

$$\widehat{\mathcal{L}} = (e^{-(\tau\wedge\sigma)} - e^{-(\tau\vee\sigma)})\,(\mathbb{I}[\sigma > \tau] - \mathbb{I}[\sigma < \tau])\,\frac{1}{N}\sum_{j=1}^{N} I_0(2\sqrt{T_j\tau}), \tag{53}$$

where each $T_j \stackrel{d}{=} \mathbf{TruncExp}(\tau \wedge \sigma, \tau \vee \sigma, 1)$ are independent random variables. Then we can define

$$\widehat{\Xi}_7 = \frac{1}{2}[1 - e^{-2\tau} I_0(2\tau)] + e^{-\sigma-\tau} I_0(2\sqrt{\sigma\tau}) + e^{-\tau}\widehat{\mathcal{L}}. \tag{54}$$

It is also easy to show this is an unbiased estimator of (2), with variance given by

$$\begin{aligned} \mathbb{W}[\widehat{\Xi}_7] &= \frac{e^{-2\tau}}{N}\left[e^{-(\tau\wedge\sigma)} - e^{-(\tau\vee\sigma)}\right]^2 \\ &\quad \times \left(\mathbb{E}[I_0^2(2\sqrt{T\tau})] - \left(\mathbb{E}[I_0(2\sqrt{T\tau})]\right)^2\right). \end{aligned} \tag{55}$$

4 Performance and Analysis of Estimators

We now consider the performance of the estimators introduced in Section 3. This will be done by considering numerical estimates, and comparing them to estimates obtained by numerical integration. In particular, we will be interested in how these estimators perform in comparison to Adaptive Simpson Quadrature (ASQ) [Lyness and Kaganove 1976]. Throughout we will use a tolerance of 10^{-8} for ASQ. Additionally, we will include some comparisons to results based upon truncated Taylor series approximations [Shnidman 1989]. We base the latter on the following Taylor series expansion, which can be found in [Schwartz, Bennett and Stein 1996, Equation A-4-7]:

$$\rho(\sigma, \tau) = 1 - e^{-(\sigma+\tau)} \sum_{m=1}^{\infty} \left(\sqrt{\frac{\tau}{\sigma}}\right)^m I_m(2\sqrt{\sigma\tau}). \tag{56}$$

One can truncate the Taylor series in (56), and use the partial sum as an approximation for $\rho(\sigma, \tau)$.

Before presenting these numerical comparisons, we firstly consider simulation gains.

4.1 Simulation Gains

Appendix C contains a number of plots of simulation gains. For the sake of brevity, we only consider a subset of the 21 possible combinations of pairs of 7 estimators. Recall that the simulation gain Γ measures the number of simulation runs one estimator needs to match the same level of variance as another estimator. Thus it can indicate whether one estimator will perform as well as another, except for less simulation runs.

To begin, we consider whether the Importance Sampling estimator $\widehat{\Xi}_2$ is an improvement on the standard Poisson estimator $\widehat{\Xi}_1$. Figure C.1 shows a plot of the simulation gain (34), with the IS estimator using $\theta = 2$. The surface shows the logarithmic gain, as a function of σ and τ. Figure C.2 shows a cross sectional view of it. In view of (34), since the surface shows that $\log(\Gamma) < 0$, we conclude that the estimator $\widehat{\Xi}_1$ will be more efficient. Also, similar such simulation gain plots, for θ increasing in the IS estimator $\widehat{\Xi}_2$, did not indicate that the IS estimator is more efficient than $\widehat{\Xi}_1$.

Figures C.3 to C.10 provide gain plots, comparing the Poisson estimator $\widehat{\Xi}_1$ to some of the estimators based upon continuous sampling distributions and the results of Theorem 1, Parts (iii) and (iv). In these plots the gain Γ is the ratio of the number of simulations used in estimators $\widehat{\Xi}_4$, $\widehat{\Xi}_5$, $\widehat{\Xi}_6$ and $\widehat{\Xi}_7$, compared to the number of simulations needed for the Poisson Estimator $\widehat{\Xi}_1$. In these cases if Γ is greater than zero on the logarithmic scale, the Poisson Estimator requires less simulation runs.

Figure C.3 compares the estimator $\widehat{\Xi}_4$ to $\widehat{\Xi}_1$, and as the cross-sectional view shows in Figure C.4, there are only small regions where $\widehat{\Xi}_4$ will be more efficient.

Figure C.5 examines the performance of $\widehat{\Xi}_5$ relative to $\widehat{\Xi}_1$, and Figure C.6 shows a cross-sectional view. As for the previous case, there are regions where the continuous estimator outperforms the Poisson estimator.

Figure C.7, in contrast to the two examples considered previously, showed significant global improvements on the estimator $\widehat{\Xi}_1$. This plot is of the gain of estimator $\widehat{\Xi}_6$ relative to $\widehat{\Xi}_1$. Both Figure C.7, and the cross-sectional plot of Figure C.8, show that the estimator $\widehat{\Xi}_6$ will frequently outperform the Poisson estimator.

The final comparison we consider is that of the simulation gain of estimator $\widehat{\Xi}_7$ relative to $\widehat{\Xi}_1$. As can be observed from Figures C.9 and C.10, there are many choices of σ and τ-parameters which will result in simulation savings in using estimator $\widehat{\Xi}_7$.

The Figures show that for each estimator there are values of σ and τ for which less simulation runs are required than for $\widehat{\Xi}_1$.

The main conclusion from these simulation gain plots is that some of the estimators based upon Theorem 1, Parts (iii) and (iv), will outperform the discrete Poisson estimator, based upon Part (ii) of Theorem 1. Estimators $\widehat{\Xi}_6$ and $\widehat{\Xi}_7$ showed the most promise, while the other continuous estimators considered also had regions where improvements over the Poisson estimator were possible. Clearly, the Importance Sampling estimator $\widehat{\Xi}_2$ had inferior performance to the Poisson estimator $\widehat{\Xi}_1$.

4.2 Numerical Results

We now consider numerical results of these estimators, and will be more interested in accuracy when compared to results obtained using ASQ. All estimates can be found in the Tables in Appendix D. Table D.1 contains some estimates of $\rho(\sigma, \tau)$ based upon a truncated Taylor series approximation using (56). The partial sum uses 100 terms to obtain the estimate. Table D.1 also contains estimates obtained using ASQ, and the absolute error between the estimates is included. This error is just the difference between the two estimates. These results will be used to compare the performance of the estimators of Section 3. We do not consider the IS estimator $\widehat{\Xi}_2$, due to the fact that its simulation gain plot showed it to be generally inferior to the Poisson estimator $\widehat{\Xi}_1$.

Table D.2 contains a selection of estimates for estimator $\widehat{\Xi}_1$. The two free parameters σ and τ range from 1 to 5, and the Table shows estimates based on samples using $N = 10^3$, 10^4, 10^5 and 10^6. Each estimate is compared to one obtained via ASQ, and the absolute error is also given. The Table shows that the estimator performs well for larger N, but still requires a larger sample size to achieve more uniform accuracy.

Table D.3 shows the performance of the standard continuous estimator $\widehat{\Xi}_3$, again with comparisons to results based on ASQ, and with N varying as in Table D.2. The errors seem consistently smaller than those obtained in Table D.2, indicating $\widehat{\Xi}_3$ is slighly more accurate. Overall, however, there is not a major improvement over $\widehat{\Xi}_1$, just a small order of magnitude improvement.

We now consider estimators based upon the stochastic representations of Theorem 1, Parts (iii) and (iv). Tables D.4 and D.5 contain estimates based upon $\widehat{\Xi}_4$ and $\widehat{\Xi}_6$. Both these estimators are based upon Part (iii) of Theorem 1, with $\widehat{\Xi}_4$ using a uniform sampling distribution, and $\widehat{\Xi}_6$ employing a Truncated Exponential sampling distribution. Table D.4 shows estimates using $N = 10^3$ and 10^4, and compares results again to those obtained via ASQ. It is clear that for the relatively modest sample sizes, these estimators are performing better than those considered in Tables D.2 and D.3. The results in Table 5 are generated for the same estimators as in Table D.4, but use $N = 10^5$ and 10^6. These show further improvements are made on the accuracy of estimators $\widehat{\Xi}_4$ and $\widehat{\Xi}_6$.

Tables D.6 and D.7 show simulation results using estimators $\widehat{\Xi}_5$ and $\widehat{\Xi}_7$, which are based upon Theorem 1, Part (iv). $\widehat{\Xi}_5$ uses a uniformly sampled distribution, while $\widehat{\Xi}_7$ is based upon a Truncated Exponential distribution. Table D.6 contains results for the case where $N = 10^3$ and 10^4, while Table D.7 contains estimates for $N = 10^5$ and 10^6. These two Tables show that both estimators are performing well, and that $\widehat{\Xi}_5$ is performing extremely well in some cases.

The final set of estimates can be found in Table D.8, which directly compares the performance of estimators $\widehat{\Xi}_4$, $\widehat{\Xi}_5$, $\widehat{\Xi}_6$ and $\widehat{\Xi}_7$. Each estimator uses a sample of size $N = 10^6$, and each result is again compared to ASQ. As can be observed, the estimators are performing well for this number of simulations, with $\widehat{\Xi}_5$ returning the smallest errors on average.

It is interesting to compare the results of Table D.8 with those in Table D.1, the latter being estimates based upon a partial sum approximation of (56). A sample size of $N = 10^6$, for each of the estimators considered in Table D.8, are not as accurate as an estimate based upon a partial sum of 100 terms. Increasing N in these estimators will improve their accuracy, but will increase computation times. It is worth noting, however, that the computation time to compute the partial sum series from (56) is faster than computing 10^6 simulation runs in a Monte Carlo estimator. This sample size issue is a significant limiting factor on the application of Monte Carlo methods.

5 Conclusions

This report is an investigation of the Monte Carlo estimation of Marcum's Q-Function. Using some new stochastic representations derived in [Weinberg 2005], together with a new result derived in this report, seven estimators were defined and analysed. Two of these estimators were based upon discrete sampling distributions. One was based upon the Poisson association in Part (ii) of Theorem 1. The second was an importance sampling estimator, using a tilted sampling distribution. It was found that the Poisson estimator $\widehat{\Xi}_1$ was the better of the two estimators. The remaining five estimators used continuous sampling distributions. One was based upon a Truncated Exponential Distribution, on the semi-infinite domain $[\tau, \infty)$. The remaining four estimators were based upon Parts (iii) and (iv) of Theorem 1. Two used Uniform sampling distributions, while the remaining pair used Truncated Exponential sampling distributions, on a finite domain. Out of all the continuous estimators, it was found that they had regions where they performed very well, and similarly regions where their performance was moderate. It was found that the most efficient of the seven estimators was the one based upon Part (iv) of Theorem 1, using a uniform sampling distribution, referred to as $\widehat{\Xi}_5$.

The performance analysis of the seven estimators indicated that large sample sizes are needed to obtain accurate results, although some of the estimators returned accurate results for relatively small sample sizes, such as $\widehat{\Xi}_5$ and $\widehat{\Xi}_7$. Simulation gain considerations indicated that a number of these estimators may be useful from a practical perspective.

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Appendix A: Some Properties of Statistical Variance

For completeness we provide a concise outline of the definitions of statistical means and variances of random variables, as well as some properties of variances used throughout this report. The interested reader is referred to [Billingsley 1995] for a rigorous treatment of the foundations of probability, while the more practically oriented reader is referred to [Durrett 1996 and Ross 2002].

Suppose $X : \Omega \longrightarrow \mathbb{R}$ is a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Its mean or expectation is defined by the the integral

$$\mathbb{E}[X] := \int_\Omega X(\omega)\mathbb{P}(d\omega). \tag{A.1}$$

In the case of an atomic measure, or equivalently, X takes discrete values, the expectation (A.1) reduces to a weighted sum of the values of X, with weights being the associated point probabilities. If the probability measure is absolutely continuous with repect to Lebesgue measure μ on the real line, then by the Radon-Nikodym Theorem, there exists a derivative $g(\omega)$, known as a density, such that $\mathbb{P}(d\omega) = g(\omega)\mu(d\omega)$. This implies (A.1) becomes

$$\mathbb{E}[X] = \int_{\mathbb{R}} X(\omega)g(\omega)\mu(d\omega). \tag{A.2}$$

Random variables with such a density are known as continuous, and (A.2) is the well-known expression for the expectation of such random variables.

The variance of a random variable X is defined to be its average squared deviation from its mean. Specifically, we can write this as

$$\mathbb{V}[X] := \mathbb{E}\left([X - \mathbb{E}[X]]^2\right). \tag{A.3}$$

By expanding out the expression in (A.3), it can be shown that

$$\mathbb{V}[X] := \mathbb{E}[X^2] - (\mathbb{E}[X])^2, \tag{A.4}$$

which is in a useful form for numerical estimation.

Suppose α, $\beta \in \Omega$ are scalar constants, and that X and Y are integrable random variables. It is not difficult to show that the statistical expectation is a linear operator, which implies that $\mathbb{E}[\alpha X + \beta Y] = \alpha\mathbb{E}[X] + \beta\mathbb{E}[Y]$. Statistical variance, on the other hand, is not a linear operator. However, it has a number of useful properties, which we now consider. The first is that the variance of a scalar multiple of a random variable is just its square times the variance of the underlying random variable:

$$\mathbb{V}[\alpha X] = \alpha^2\mathbb{V}[X]. \tag{A.5}$$

Another interesting fact is that a constant added to a random variable contributes nothing to its variation from its mean:

$$\mathbb{V}[X + \alpha] = \mathbb{V}[X]. \tag{A.6}$$

This property has been used in the calculation of the variances of estimators based upon Theorem 1, Parts (iii) and (iv).

The final property of statistical variance that we consider provides an expression for the variance of a sum of two random variables:

$$\mathbb{V}[X + Y] = \mathbb{V}[X] + 2\mathbb{C}[X, Y] + \mathbb{V}[Y], \tag{A.7}$$

where $\mathbb{C}[X, Y] := \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$ is known as the covariance of X and Y. It gives a measure of the association between the two random variables. In the case where these random variables are independent, it can be shown that $\mathbb{C}[X, Y] = 0$ and consequently the variance of the two in (A.7) reduces to the sum of the two respective variances. This property has been used extensively in the report, since the Monte Carlo estimators are sums of independent random variables.

Appendix B: Generation of Realisations of Random Variables

We provide some notes on the generation of realisations of both discrete and continuous random variables, since this is critical to the Monte Carlo sampling approach to estimation. An excellent guide to simulation is [Ross 2002], where both the theory of simulation and practical algorithms are considered. In particular, Chapter 4 of [Ross 2002] contains an extensive overview of the techniques of generating discrete random variables, while Chapter 5 deals with the continuous case.

Most of the basic algorithms for simulation of random variables operate by transforming a random number in the unit interval $[0, 1]$ to a realisation of the given random variable. The reason for this is that it is relatively easy numerically to generate a random sample of numbers between 0 and 1. To illustrate this, one can use the digits of π to generate such a series of random numbers.

Suppose we have a discrete random variable X, which takes values x_j with point probabilities p_j for $j \in \{0, 1, 2, \ldots m\}$. Then we can generate a realisation of X using the following algorithm, known as the Inverse Transform Method:

1. Generate a random number $r \in [0, 1]$;
2. If $r < p_0$ then x_0 is the realisation, and stop, else
3. If $r < p_0 + p_1$ then x_1 is the realisation, and stop, else
4. If $r < p_0 + p_1 + p_2$ then x_2 is the realisation, and stop, else continue.

This method is examined extensively in [Ross 2002], to which the reader is referred. The only discrete Monte Carlo estimators considered in this report were based upon Poisson sampling distributions. Section 4.2 of [Ross 2002] provides an algorithm exploiting some of the properties of this distribution. We assume $X \stackrel{d}{=} \mathbf{Po}(\lambda)$, and that $r \in [0, 1]$ is a random number. The following algorithm has been taken from [Ross 2002]:

1. Set $i = 0$, $p = e^{-\lambda}, F = p$;
2. If $r < F$, set $x = i$ and stop, else
3. $p = \frac{\lambda p}{i+1}$, $F := F + p$, $i := i + 1$ and return to Step 2.

When the algorithm has finished running, the number x will be a realisation of the Poisson random variable. It can be shown that the average number of runs of this algorithm is approximately $1 + 0.798\sqrt{\lambda}$ [see Ross 2002].

A number of other algorithms are considered in [Ross 2002], to which the reader is referrred.

The Inverse Transform Algorithm is actually based upon the following result, which we state in terms of continuous random variables:

Lemma B.1 *If X is a continuous random variable with cumulative distribution function F_X, and $R \stackrel{d}{=} \mathbf{R}[0,1]$ then*

$$F_X^{-1}(R) \stackrel{d}{=} X. \tag{B.1}$$

The proof of Lemma B.1 is relatively simple, and can be found in [Ross 2002]. This means that a continuous random variable can be simulated by inverting its cumulative distribution function, and evaluating it at a random number in the unit interval $[0,1]$. For the Monte Carlo estimators considered in this report, inverting the cumulative distribution function of the sampling distribution is relatively easy. Specifically, the only estimators where this has been necessary to do have used Truncated Exponential sampling distributions. The latter have easily inverted cumulative distribution functions.

The last result we present is a useful property of uniform random numbers in the interval $[0,1]$:

Lemma B.2 *If $R \stackrel{d}{=} \mathbf{R}[0,1]$ then $1-R \stackrel{d}{=} \mathbf{R}[0,1]$.*

This is an obvious result, but we provide a short proof for the interested reader. We remark that this is frequently used in conjunction with Lemma B.1 in the generation of realisations of continuous random variables.

To prove Lemma B.2, let $Z = 1-R$ and $z \in [0,1]$. Then observe that

$$\begin{aligned} \mathbb{P}[Z \le z] &= \mathbb{P}[1-R \le z] \\ &= \mathbb{P}[R \ge 1-z] \\ &= 1-\mathbb{P}[R < 1-z], \end{aligned}$$

and since $1-z \in [0,1]$ also we note that

$$\begin{aligned} \mathbb{P}[Z \le z] &= 1-(1-z) \\ &= z, \end{aligned}$$

implying Z has the same distribution function as R.

Appendix C: Simulation Gains

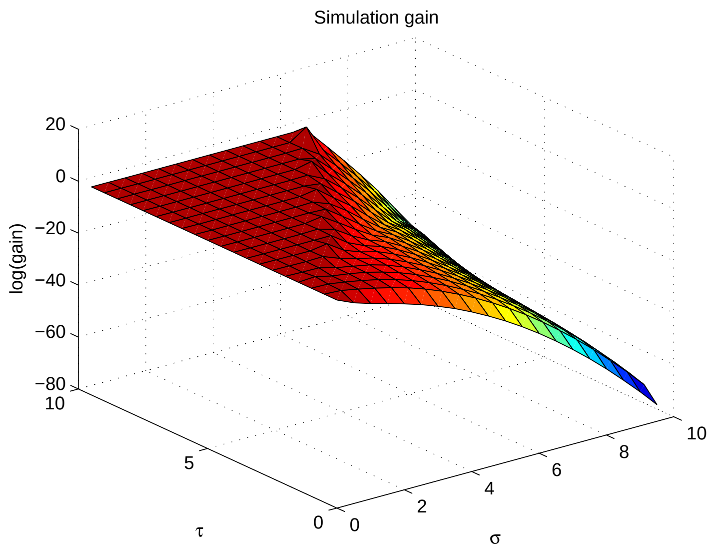

Figure C.1: The simulation gain (26) as a surface in 3-space, with the gain measured in a logarithmic scale of Poisson estimator $\widehat{\widehat{\Xi}}_1$versus estimator $\widehat{\widehat{\Xi}}_2$

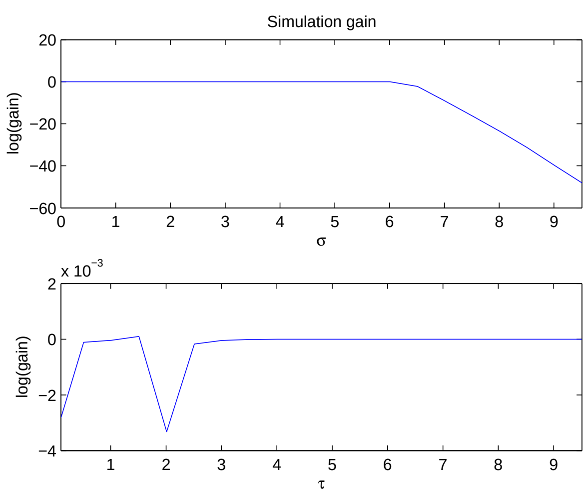

Figure C.2: Two cross sectional views of the logarithmic gain in Figure 1. The first subplot shows the gain as a function of σ, with $\tau = 20$, while the second subplot is the gain as a function of τ, with $\sigma = 1$.

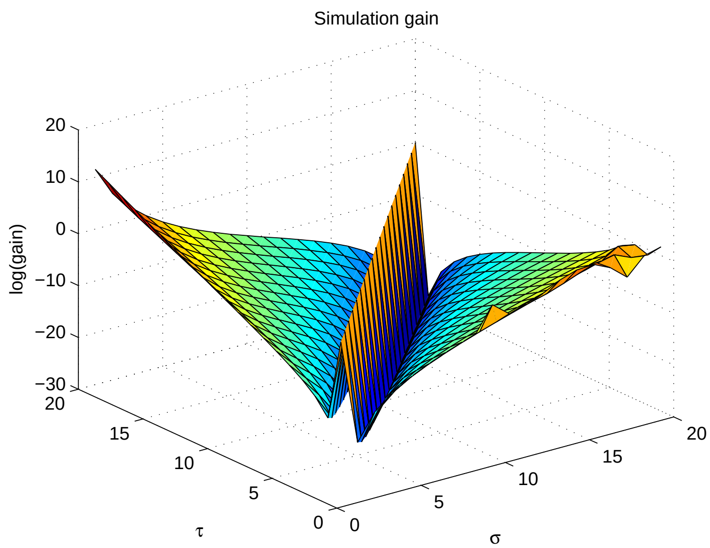

Figure C.3: The simulation gain (26) as a surface in 3-space, with the gain measured in a logarithmic scale of estimator $\widehat{\Xi}_4$ *versus Poisson estimator* $\widehat{\Xi}_1$

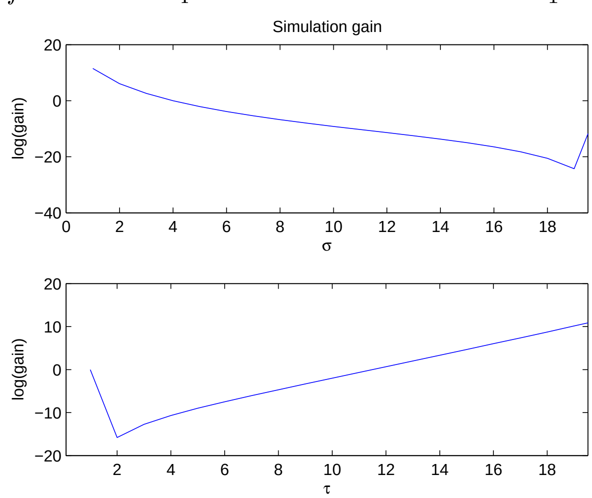

Figure C.4: Two cross sectional views of the logarithmic gain in Figure 3. The first subplot shows the gain as a function of σ*, with* $\tau = 20$*, while the second subplot is the gain as a function of* τ*, with* $\sigma = 20$*.*

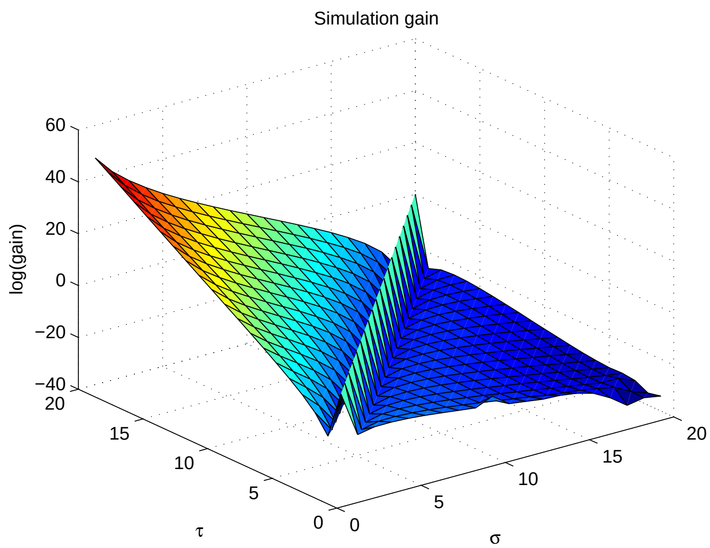

Figure C.5: The simulation gain (26) as a surface in 3-space, with the gain measured in a logarithmic scale of estimator $\widehat{\widehat{\Xi}}_5$ versus Poisson estimator $\widehat{\widehat{\Xi}}_1$

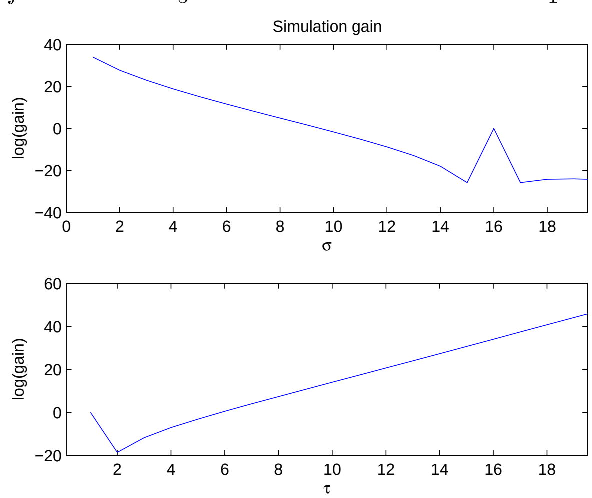

Figure C.6: Two cross sectional views of the logarithmic gain in Figure 5. The first subplot shows the gain as a function of σ, with $\tau = 16$, while the second subplot is the gain as a function of τ, with $\sigma = 1$.

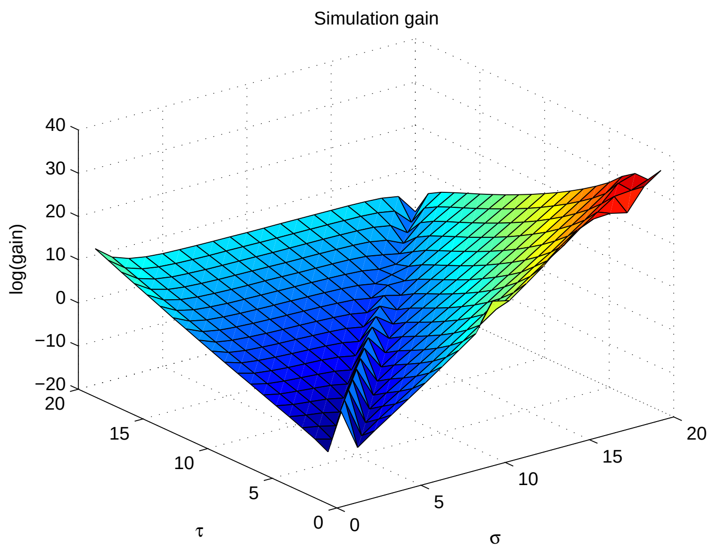

Figure C.7: The simulation gain (26) as a surface in 3-space, with the gain measured in a logarithmic scale of estimator $\widehat{\Xi}_6$ *versus Poisson estimator* $\widehat{\Xi}_1$

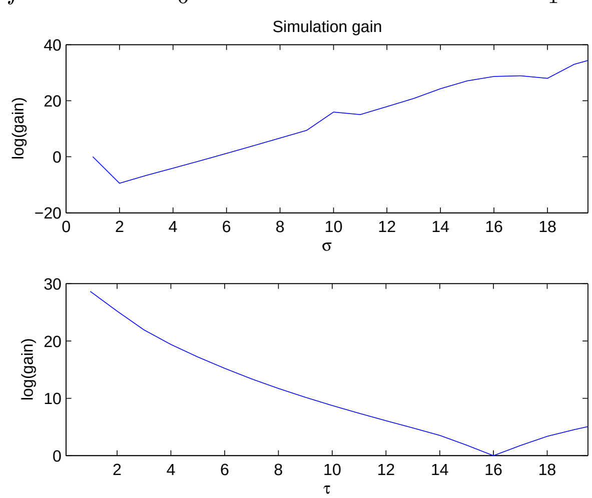

Figure C.8: Two cross sectional views of the logarithmic gain in Figure 7. The first subplot shows the gain as a function of σ*, with* $\tau = 16$*, while the second subplot is the gain as a function of* τ*, with* $\sigma = 1$*.*

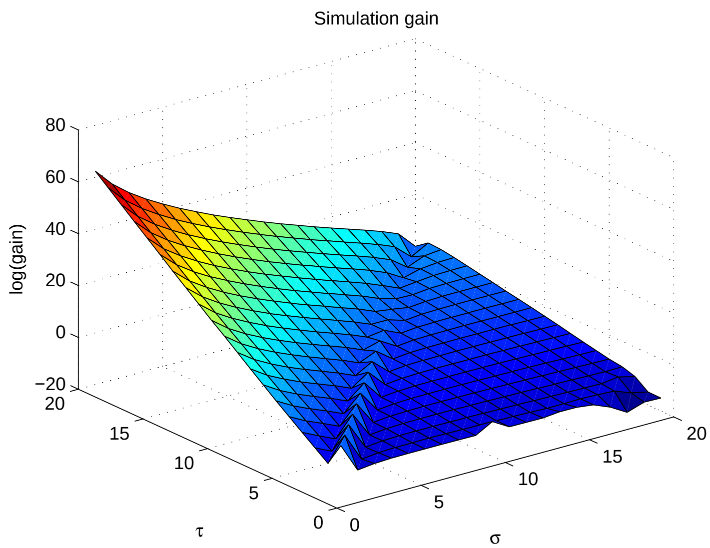

Figure C.9: The simulation gain (26) as a surface in 3-space, with the gain measured in a logarithmic scale of estimator $\widehat{\Xi}_7$ versus Poisson estimator $\widehat{\Xi}_1$

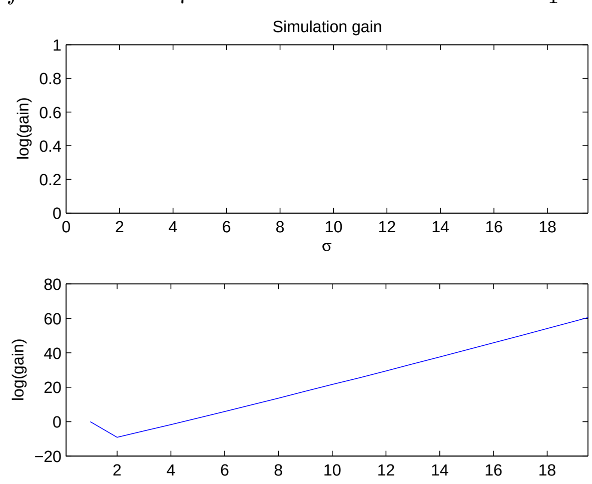

Figure C.10: Two cross sectional views of the logarithmic gain in Figure 9. The first subplot shows the gain as a function of σ, with $\tau = 16$, while the second subplot is the gain as a function of τ, with $\sigma = 1$.

Appendix D: Tables of Numerical Results

Table D.1: A selection of estimates of $\rho(\sigma,\tau)$, based on a partial sum of 100 terms using Equation (56). For each (σ,τ) pair, an estimate is compared to one obtained by Adaptive Simpson Quadrature (ASQ), with a tolerance of 10^{-8}. ϵ_1 is the absolute error between the ASQ estimate, and that based upon Equation (56).

σ	τ	ASQ	Estimate using Equation (56)	ϵ_1
1.00	1.00	6.54254161276836e-001	6.54254161276836e-001	0.0000e+000
1.00	2.00	3.94296858899549e-001	3.94296858892332e-001	7.2174e-012
1.00	3.00	2.24984708801310e-001	2.24984708790304e-001	1.1006e-011
1.00	4.00	1.23381447904444e-001	1.23381447854823e-001	4.9621e-011
1.00	5.00	6.56319493789875e-002	6.56319492124853e-002	1.6650e-010
2.00	1.00	8.17415225126923e-001	8.17415225069612e-001	5.7311e-011
2.00	2.00	6.03500960611993e-001	6.03500960611993e-001	-1.1102e-016
2.00	3.00	4.14710585222029e-001	4.14710585234130e-001	-1.2101e-011
2.00	4.00	2.70039453942757e-001	2.70039453948642e-001	-5.8854e-012
2.00	5.00	1.68568913522749e-001	1.68568913530132e-001	-7.3827e-012
3.00	1.00	9.06136886710340e-001	9.06136886583505e-001	1.2684e-010
3.00	2.00	7.53011300651773e-001	7.53011300627772e-001	2.4001e-011
3.00	3.00	5.83328716319908e-001	5.83328716319908e-001	2.2204e-016
3.00	4.00	4.26907556449231e-001	4.26907556460672e-001	-1.1441e-011
3.00	5.00	2.98193396308125e-001	2.98193396374000e-001	-6.5875e-011
4.00	1.00	9.52770303245878e-001	9.52770303246472e-001	-5.9441e-013
4.00	2.00	8.51936356981248e-001	8.51936356942411e-001	3.8837e-011
4.00	3.00	7.16950482726697e-001	7.16950482718797e-001	7.8992e-012
4.00	4.00	5.71715890928425e-001	5.71715890928425e-001	1.1102e-016
4.00	5.00	4.35072015844931e-001	4.35072015850586e-001	-5.6550e-012
5.00	1.00	9.76650054658845e-001	9.76650054770644e-001	-1.1180e-010
5.00	2.00	9.13934477595961e-001	9.13934477600213e-001	-4.2516e-012
5.00	3.00	8.14938772496419e-001	8.14938772486556e-001	9.8631e-012
5.00	4.00	6.92981835299699e-001	6.92981835296982e-001	2.7168e-012
5.00	5.00	5.63916668581714e-001	5.63916668581714e-001	0.0000e+000

Table D.2: Estimates of $\rho(\sigma,\tau)$, based upon the estimator $\widehat{\widehat{\Xi}}_1$. For each (σ,τ) pair, an estimate is compared to one obtained by ASQ, with a tolerance of 10^{-8}. ϵ_1 is the absolute error between the exact result and $\widehat{\widehat{\Xi}}_1$.

σ	τ	ASQ	$N = 10^3$ $\widehat{\widehat{\Xi}}_1$	ϵ_1	$N = 10^4$ $\widehat{\widehat{\Xi}}_1$	ϵ_1
1.00	1.00	6.54254161276836e-001	6.59663019936582e-001	-5.4089e-003	6.50360378055821e-001	3.8938e-003
1.00	2.00	3.94296858899549e-001	3.85194290598783e-001	9.1026e-003	3.92362850179273e-001	1.9340e-003
1.00	3.00	2.24984708801310e-001	2.17743743506852e-001	7.2410e-003	2.26391570580864e-001	-1.4069e-003
1.00	4.00	1.23381447904444e-001	1.30685746598897e-001	-7.3043e-003	1.24564904290777e-001	-1.1835e-003
1.00	5.00	6.56319493789875e-002	6.40133039692288e-002	1.6186e-003	6.58733516713669e-002	-2.4140e-004
2.00	1.00	8.17415225126923e-001	8.08786922481781e-001	8.6283e-003	8.16190325312452e-001	1.2249e-003
2.00	2.00	6.03500960611993e-001	5.99414147437010e-001	4.0868e-003	5.99124590079898e-001	4.3764e-003
2.00	3.00	4.14710585222029e-001	4.12974008074643e-001	1.7366e-003	4.17823901762987e-001	-3.1133e-003
2.00	4.00	2.70039453942757e-001	2.87372723033554e-001	-1.7333e-002	2.68488904729298e-001	1.5505e-003
2.00	5.00	1.68568913522749e-001	1.66041880977511e-001	2.5270e-003	1.71171430841358e-001	-2.6025e-003
3.00	1.00	9.06136886710340e-001	9.03764463728220e-001	2.3724e-003	9.04907858216395e-001	1.2290e-003
3.00	2.00	7.53011300651773e-001	7.57008385124267e-001	-3.9971e-003	7.49426192255081e-001	3.5851e-003
3.00	3.00	5.83328716319908e-001	5.91643315593691e-001	-8.3146e-003	5.81100102494980e-001	2.2286e-003
3.00	4.00	4.26907556449231e-001	4.09232406814433e-001	1.7675e-002	4.30487426966499e-001	-3.5799e-003
3.00	5.00	2.98193396308125e-001	2.96272130229952e-001	1.9213e-003	2.98006067142840e-001	1.8733e-004
4.00	1.00	9.52770303245878e-001	9.52507276019420e-001	2.6303e-004	9.50171195028357e-001	2.5991e-003
4.00	2.00	8.51936356981248e-001	8.51822728323425e-001	1.1363e-004	8.53790401511696e-001	-1.8540e-003
4.00	3.00	7.16950482726697e-001	7.13097900186227e-001	3.8526e-003	7.17287156477901e-001	-3.3667e-004
4.00	4.00	5.71715890928425e-001	5.59632865997872e-001	1.2083e-002	5.74210225421900e-001	-2.4943e-003
4.00	5.00	4.35072015844931e-001	4.23091200594429e-001	1.1981e-002	4.30558437079540e-001	4.5136e-003
5.00	1.00	9.76650054658845e-001	9.77959383696456e-001	-1.3093e-003	9.76492236079186e-001	1.5782e-004
5.00	2.00	9.13934477595961e-001	9.23956427259091e-001	-1.0022e-002	9.12224012645041e-001	1.7105e-003
5.00	3.00	8.14938772496419e-001	8.17799270221826e-001	-2.8605e-003	8.19316112259624e-001	-4.3773e-003
5.00	4.00	6.92981835299699e-001	7.09843084153854e-001	-1.6861e-002	6.85342014730607e-001	7.6398e-003
5.00	5.00	5.63916668581714e-001	5.68704289502493e-001	-4.7876e-003	5.65107395707819e-001	-1.1907e-003

σ	τ	ASQ	$N = 10^5$ $\widehat{\widehat{\Xi}}_1$	ϵ_1	$N = 10^6$ $\widehat{\widehat{\Xi}}_1$	ϵ_1
1.00	1.00	6.54254161276836e-001	6.53782077383129e-001	4.7208e-004	6.54330711738798e-001	-7.6550e-005
1.00	2.00	3.94296858899549e-001	3.94392475570467e-001	-9.5617e-005	3.94434963977153e-001	-1.3811e-004
1.00	3.00	2.24984708801310e-001	2.24678879648360e-001	3.0583e-004	2.25374718271147e-001	-3.9001e-004
1.00	4.00	1.23381447904444e-001	1.22795178392011e-001	5.8627e-004	1.23161681422234e-001	2.1977e-004
1.00	5.00	6.56319493789875e-002	6.56362957143730e-002	-4.3463e-006	6.56172255656532e-002	1.4724e-005
2.00	1.00	8.17415225126923e-001	8.17480136368409e-001	-6.4911e-005	8.17248105021029e-001	1.6712e-004
2.00	2.00	6.03500960611993e-001	6.04135576691294e-001	-6.3462e-004	6.03134593656892e-001	3.6637e-004
2.00	3.00	4.14710585222029e-001	4.14206316640914e-001	5.0427e-004	4.14698457454405e-001	1.2128e-005
2.00	4.00	2.70039453942757e-001	2.70776906458255e-001	-7.3745e-004	2.69970795292826e-001	6.8659e-005
2.00	5.00	1.68568913522749e-001	1.69365858333398e-001	-7.9694e-004	1.68718140430055e-001	-1.4923e-004
3.00	1.00	9.06136886710340e-001	9.06018830365911e-001	1.1806e-004	9.06183146646901e-001	-4.6260e-005
3.00	2.00	7.53011300651773e-001	7.53905328428387e-001	-8.9403e-004	7.52755663345111e-001	2.5564e-004
3.00	3.00	5.83328716319908e-001	5.84539256814320e-001	-1.2105e-003	5.83258987927966e-001	6.9728e-005
3.00	4.00	4.26907556449231e-001	4.26570791546415e-001	3.3676e-004	4.27102889049943e-001	-1.9533e-004
3.00	5.00	2.98193396308125e-001	2.98434620073621e-001	-2.4122e-004	2.98038165920483e-001	1.5523e-004
4.00	1.00	9.52770303245878e-001	9.52433042857598e-001	3.3726e-004	9.52855908650958e-001	-8.5605e-005
4.00	2.00	8.51936356981248e-001	8.51631242117646e-001	3.0511e-004	8.51895133387747e-001	4.1224e-005
4.00	3.00	7.16950482726697e-001	7.15850360579888e-001	1.1001e-003	7.16728656850178e-001	2.2183e-004
4.00	4.00	5.71715890928425e-001	5.72715698209388e-001	-9.9981e-004	5.72184483654959e-001	-4.6859e-004
4.00	5.00	4.35072015844931e-001	4.36055921252495e-001	-9.8391e-004	4.35513104967410e-001	-4.4109e-004
5.00	1.00	9.76650054658845e-001	9.76506652946475e-001	1.4340e-004	9.76700032704114e-001	-4.9978e-005
5.00	2.00	9.13934477595961e-001	9.14728901446874e-001	-7.9442e-004	9.14001592767651e-001	-6.7115e-005
5.00	3.00	8.14938772496419e-001	8.15676367093696e-001	-7.3759e-004	8.15042078532561e-001	-1.0331e-004
5.00	4.00	6.92981835299699e-001	6.92108253233747e-001	8.7358e-004	6.93350564796337e-001	-3.6873e-004
5.00	5.00	5.63916668581714e-001	5.63164165210903e-001	7.5250e-004	5.64265650772492e-001	-3.4898e-004

Table D.3: Estimates of $\rho(\sigma,\tau)$, based upon the estimator $\widehat{\Xi}_3$. For each (σ,τ) pair, an estimate is compared to one obtained by ASQ, with a tolerance of 10^{-8}. ϵ_1 is the absolute error between the ASQ estimate and $\widehat{\Xi}_3$.

σ	τ	ASQ	$N = 10^3$ $\widehat{\Xi}_3$	ϵ_1	$N = 10^4$ $\widehat{\Xi}_3$	ϵ_1
1.00	1.00	6.54254161276836e-001	6.42351791738002e-001	1.1902e-002	6.47813986346734e-001	6.4402e-003
1.00	2.00	3.94296858899549e-001	3.84581796920965e-001	9.7151e-003	3.95055852211444e-001	-7.5899e-004
1.00	3.00	2.24984708801310e-001	2.23945975009446e-001	1.0387e-003	2.25765300947652e-001	-7.8059e-004
1.00	4.00	1.23381447904444e-001	1.27490629537926e-001	-4.1092e-003	1.22895779602500e-001	4.8567e-004
1.00	5.00	6.56319493789875e-002	6.39595113545533e-002	1.6724e-003	6.58885895504813e-002	-2.5664e-004
2.00	1.00	8.17415225126923e-001	8.48389410777061e-001	-3.0974e-002	8.02840130950167e-001	1.4575e-002
2.00	2.00	6.03500960611993e-001	6.22660876505515e-001	-1.9160e-002	5.91721898338978e-001	1.1779e-002
2.00	3.00	4.14710585222029e-001	3.91210406341459e-001	2.3500e-002	4.12100036069522e-001	2.6105e-003
2.00	4.00	2.70039453942757e-001	2.67333456505119e-001	2.7060e-003	2.67843599430930e-001	2.1959e-003
2.00	5.00	1.68568913522749e-001	1.61550881730203e-001	7.0180e-003	1.69613775897855e-001	-1.0449e-003
3.00	1.00	9.06136886710340e-001	7.88022736235007e-001	1.1811e-001	9.10184840635533e-001	-4.0480e-003
3.00	2.00	7.53011300651773e-001	6.69025867145190e-001	8.3985e-002	7.14221776509817e-001	3.8790e-002
3.00	3.00	5.83328716319908e-001	5.57420230430789e-001	2.5908e-002	5.87272503477971e-001	-3.9438e-003
3.00	4.00	4.26907556449231e-001	4.23768363216703e-001	3.1392e-003	4.22173707770064e-001	4.7338e-003
3.00	5.00	2.98193396308125e-001	3.54706792528251e-001	-5.6513e-002	2.99752426131303e-001	-1.5590e-003
4.00	1.00	9.52770303245878e-001	9.42441640335310e-001	1.0329e-002	8.38828291254819e-001	1.1394e-001
4.00	2.00	8.51936356981248e-001	7.42459380843225e-001	1.0948e-001	8.43738914198135e-001	8.1974e-003
4.00	3.00	7.16950482726697e-001	6.29249481147251e-001	8.7701e-002	7.33637956665969e-001	-1.6687e-002
4.00	4.00	5.71715890928425e-001	5.73960509726093e-001	-2.2446e-003	6.14784521432351e-001	-4.3069e-002
4.00	5.00	4.35072015844931e-001	3.74332332071409e-001	6.0740e-002	4.56334092417138e-001	-2.1262e-002
5.00	1.00	9.76650054658845e-001	1.82160744307321e+000	-8.4496e-001	1.04564373266554e+000	-6.8994e-002
5.00	2.00	9.13934477595961e-001	1.35808204532641e+000	-4.4415e-001	9.86783853314269e-001	-7.2849e-002
5.00	3.00	8.14938772496419e-001	1.03824213755018e+000	-2.2330e-001	8.03249822925602e-001	1.1689e-002
5.00	4.00	6.92981835299699e-001	6.44248629605388e-001	4.8733e-002	7.01817499213763e-001	-8.8357e-003
5.00	5.00	5.63916668581714e-001	4.86413080005933e-001	7.7504e-002	5.76199915419100e-001	-1.2283e-002

σ	τ	ASQ	$N = 10^5$ $\widehat{\Xi}_3$	ϵ_1	$N = 10^6$ $\widehat{\Xi}_3$	ϵ_1
1.00	1.00	6.54254161276836e-001	6.54040572750096e-001	2.1359e-004	6.54054793604104e-001	1.9937e-004
1.00	2.00	3.94296858899549e-001	3.94480768124019e-001	-1.8391e-004	3.94700830230529e-001	-4.0397e-004
1.00	3.00	2.24984708801310e-001	2.25088605527676e-001	-1.0390e-004	2.24780014304186e-001	2.0469e-004
1.00	4.00	1.23381447904444e-001	1.23354460768421e-001	2.6987e-005	1.23487855731148e-001	-1.0641e-004
1.00	5.00	6.56319493789875e-002	6.57255937823738e-002	-9.3644e-005	6.56356163389027e-002	-3.6670e-006
2.00	1.00	8.17415225126923e-001	8.13101167357929e-001	4.3141e-003	8.17058974458540e-001	3.5625e-004
2.00	2.00	6.03500960611993e-001	6.03530267194846e-001	-2.9307e-005	6.03663028146881e-001	-1.6207e-004
2.00	3.00	4.14710585222029e-001	4.14085627724929e-001	6.2496e-004	4.14618831126164e-001	9.1754e-005
2.00	4.00	2.70039453942757e-001	2.69730354883825e-001	3.0910e-004	2.70611450811507e-001	-5.7200e-004
2.00	5.00	1.68568913522749e-001	1.69058964500188e-001	-4.9005e-004	1.68407891936721e-001	1.6102e-004
3.00	1.00	9.06136886710340e-001	9.02097692305598e-001	4.0392e-003	8.96295234852391e-001	9.8417e-003
3.00	2.00	7.53011300651773e-001	7.46480080922404e-001	6.5312e-003	7.51461245093084e-001	1.5501e-003
3.00	3.00	5.83328716319908e-001	5.79016367117777e-001	4.3123e-003	5.83665782360501e-001	-3.3707e-004
3.00	4.00	4.26907556449231e-001	4.23438123445503e-001	3.4694e-003	4.28286131002742e-001	-1.3786e-003
3.00	5.00	2.98193396308125e-001	3.00914377642216e-001	-2.7210e-003	2.98471672558105e-001	-2.7828e-004
4.00	1.00	9.52770303245878e-001	9.66625883042089e-001	-1.3856e-002	9.46314040220739e-001	6.4563e-003
4.00	2.00	8.51936356981248e-001	8.21055664707472e-001	3.0881e-002	8.60765986551164e-001	-8.8296e-003
4.00	3.00	7.16950482726697e-001	7.14543879762383e-001	2.4066e-003	7.16205642122862e-001	7.4484e-004
4.00	4.00	5.71715890928425e-001	5.69415946836832e-001	2.2999e-003	5.72197277563860e-001	-4.8139e-004
4.00	5.00	4.35072015844931e-001	4.37475971268682e-001	-2.4040e-003	4.31889095683744e-001	3.1829e-003
5.00	1.00	9.76650054658845e-001	9.47719137363388e-001	2.8931e-002	9.73953446195458e-001	2.6966e-003
5.00	2.00	9.13934477595961e-001	8.71030178437251e-001	4.2904e-002	9.14222131583857e-001	-2.8765e-004
5.00	3.00	8.14938772496419e-001	7.90588059645628e-001	2.4351e-002	8.03082542159992e-001	1.1856e-002
5.00	4.00	6.92981835299699e-001	6.93211245758518e-001	-2.2941e-004	6.91548097188014e-001	1.4337e-003
5.00	5.00	5.63916668581714e-001	5.76094358409922e-001	-1.2178e-002	5.64368323108793e-001	-4.5165e-004

Table D.4: A selection of estimates of $\rho(\sigma, \tau)$, based upon the estimators $\widehat{\Xi}_4$ and $\widehat{\Xi}_6$. For each (σ, τ) pair, an estimate is compared to one obtained by ASQ, with a tolerance of 10^{-8}. Both ϵ_1 and ϵ_2 are the absolute error between the ASQ estimate and $\widehat{\Xi}_4$ and $\widehat{\Xi}_6$ respectively. The first half of the table sets $N = 10^3$ while the second sets $N = 10^4$.

σ	τ	ASQ	$\widehat{\Xi}_4$	ϵ_1	$\widehat{\Xi}_6$	ϵ_2
1.00	1.00	6.54254161276836e-001	6.54254161276836e-001	0.0000e+000	6.54254161276836e-001	0.0000e+000
1.00	2.00	3.94296858899549e-001	3.93105809320516e-001	1.1910e-003	3.94237101841408e-001	5.9757e-005
1.00	3.00	2.24984708801310e-001	2.21679512438498e-001	3.3052e-003	2.23010390592652e-001	1.9743e-003
1.00	4.00	1.23381447904444e-001	1.23120821402061e-001	2.6063e-004	1.19680878137504e-001	3.7006e-003
1.00	5.00	6.56319493789875e-002	6.74948004042000e-002	-1.8629e-003	3.48652869148997e-002	3.0767e-002
2.00	1.00	8.17415225126923e-001	8.17522174185000e-001	-1.0695e-004	8.16810694255860e-001	6.0453e-004
2.00	2.00	6.03500960611993e-001	6.03500960611993e-001	0.0000e+000	6.03500960611993e-001	0.0000e+000
2.00	3.00	4.14710585222029e-001	4.15061513519558e-001	-3.5093e-004	4.13229078967856e-001	1.4815e-003
2.00	4.00	2.70039453942757e-001	2.66871137161905e-001	3.1683e-003	2.65199467166843e-001	4.8400e-003
2.00	5.00	1.68568913522749e-001	1.64444988948738e-001	4.1239e-003	1.59395887758102e-001	9.1730e-003
3.00	1.00	9.06136886710340e-001	9.06917615676326e-001	-7.8073e-004	9.08324740882525e-001	-2.1879e-003
3.00	2.00	7.53011300651773e-001	7.53033618186277e-001	-2.2318e-005	7.55031147188696e-001	-2.0198e-003
3.00	3.00	5.83328716319908e-001	5.83328716319908e-001	0.0000e+000	5.83328716319908e-001	0.0000e+000
3.00	4.00	4.26907556449231e-001	4.26728306950872e-001	1.7925e-004	4.26409162407343e-001	4.9839e-004
3.00	5.00	2.98193396308125e-001	2.97253416315856e-001	9.3998e-004	3.01259321187718e-001	-3.0659e-003
4.00	1.00	9.52770303245878e-001	9.54292929510070e-001	-1.5226e-003	9.54729537014618e-001	-1.9592e-003
4.00	2.00	8.51936356981248e-001	8.52332539751002e-001	-3.9618e-004	8.47323364392281e-001	4.6130e-003
4.00	3.00	7.16950482726697e-001	7.16962505542828e-001	-1.2023e-005	7.17454008587237e-001	-5.0353e-004
4.00	4.00	5.71715890928425e-001	5.71715890928425e-001	0.0000e+000	5.71715890928425e-001	0.0000e+000
4.00	5.00	4.35072015844931e-001	4.35019406163265e-001	5.2610e-005	4.35445537648950e-001	-3.7352e-004
5.00	1.00	9.76650054658845e-001	9.78089443478190e-001	-1.4394e-003	9.66346180387962e-001	1.0304e-002
5.00	2.00	9.13934477595961e-001	9.14348322670781e-001	-4.1385e-004	9.10870843158197e-001	3.0636e-003
5.00	3.00	8.14938772496419e-001	8.14619038158618e-001	3.1973e-004	8.18878432314698e-001	-3.9397e-003
5.00	4.00	6.92981835299699e-001	6.92992478392866e-001	-1.0643e-005	6.92529565690539e-001	4.5227e-004
5.00	5.00	5.63916668581714e-001	5.63916668581714e-001	0.0000e+000	5.63916668581714e-001	0.0000e+000
1.00	1.00	6.54254161276836e-001	6.54254161276836e-001	0.0000e+000	6.54254161276836e-001	0.0000e+000
1.00	2.00	3.94296858899549e-001	3.94413745865224e-001	-1.1689e-004	3.94894322303876e-001	-5.9746e-004
1.00	3.00	2.24984708801310e-001	2.22669210477421e-001	2.3155e-003	2.26711144930319e-001	-1.7264e-003
1.00	4.00	1.23381447904444e-001	1.24197104657575e-001	-8.1566e-004	1.24282815235069e-001	-9.0137e-004
1.00	5.00	6.56319493789875e-002	6.57191933427541e-002	-8.7244e-005	6.80993077477339e-002	-2.4674e-003
2.00	1.00	8.17415225126923e-001	8.17414742633396e-001	4.8249e-007	8.17114265564211e-001	3.0096e-004
2.00	2.00	6.03500960611993e-001	6.03500960611993e-001	0.0000e+000	6.03500960611993e-001	0.0000e+000
2.00	3.00	4.14710585222029e-001	4.14716356084739e-001	-5.7709e-006	4.13784857542772e-001	9.2573e-004
2.00	4.00	2.70039453942757e-001	2.69942684229319e-001	9.6770e-005	2.69451183341462e-001	5.8827e-004
2.00	5.00	1.68568913522749e-001	1.68793327945998e-001	-2.2441e-004	1.68166142040701e-001	4.0277e-004
3.00	1.00	9.06136886710340e-001	9.06085378574383e-001	5.1508e-005	9.06260210495275e-001	-1.2332e-004
3.00	2.00	7.53011300651773e-001	7.53014194116136e-001	-2.8935e-006	7.52365982382605e-001	6.4532e-004
3.00	3.00	5.83328716319908e-001	5.83328716319908e-001	0.0000e+000	5.83328716319908e-001	0.0000e+000
3.00	4.00	4.26907556449231e-001	4.26813709927285e-001	9.3847e-005	4.27142266960049e-001	-2.3471e-004
3.00	5.00	2.98193396308125e-001	2.98345040185542e-001	-1.5164e-004	2.96844299653695e-001	1.3491e-003
4.00	1.00	9.52770303245878e-001	9.53146239774615e-001	-3.7594e-004	9.52658391696258e-001	1.1191e-004
4.00	2.00	8.51936356981248e-001	8.51906925135821e-001	2.9432e-005	8.49202036473870e-001	2.7343e-003
4.00	3.00	7.16950482726697e-001	7.16951804546839e-001	-1.3218e-006	7.16885326562727e-001	6.5156e-005
4.00	4.00	5.71715890928425e-001	5.71715890928425e-001	0.0000e+000	5.71715890928425e-001	0.0000e+000
4.00	5.00	4.35072015844931e-001	4.34983258330839e-001	8.8758e-005	4.34925131254121e-001	1.4688e-004
5.00	1.00	9.76650054658845e-001	9.76182480715274e-001	4.6757e-004	9.65865677054046e-001	1.0784e-002
5.00	2.00	9.13934477595961e-001	9.14791878942523e-001	-8.5740e-004	9.09254254401868e-001	4.6802e-003
5.00	3.00	8.14938772496419e-001	8.14910653148305e-001	2.8119e-005	8.14315445697836e-001	6.2333e-004
5.00	4.00	6.92981835299699e-001	6.92966654858981e-001	1.5180e-005	6.92938201363760e-001	4.3634e-005
5.00	5.00	5.63916668581714e-001	5.63916668581714e-001	0.0000e+000	5.63916668581714e-001	0.0000e+000

Table D.5: Based upon the estimators $\widehat{\Xi}_4$ and $\widehat{\Xi}_6$, a selection of estimates of $\rho(\sigma, \tau)$. For each (σ, τ) pair, an estimate is compared to one obtained by ASQ, with a tolerance of 10^{-8}. Both ϵ_1 and ϵ_2 are the absolute error between the ASQ estimate and $\widehat{\Xi}_4$ and $\widehat{\Xi}_6$ respectively. The first half of the table sets $N = 10^5$, second sets $N = 10^6$.

σ	τ	ASQ	$\widehat{\Xi}_4$	ϵ_1	$\widehat{\Xi}_6$	ϵ_2
1.00	1.00	6.54254161276836e-001	6.54254161276836e-001	0.0000e+000	6.54254161276836e-001	0.0000e+000
1.00	2.00	3.94296858899549e-001	3.94220483068006e-001	7.6376e-005	3.94142488451490e-001	1.5437e-004
1.00	3.00	2.24984708801310e-001	2.24716522906453e-001	2.6819e-004	2.24168819966970e-001	8.1589e-004
1.00	4.00	1.23381447904444e-001	1.22846311839826e-001	5.3514e-004	1.23513954228054e-001	-1.3251e-004
1.00	5.00	6.56319493789875e-002	6.70121471931229e-002	-1.3802e-003	6.39721344035294e-002	1.6598e-003
2.00	1.00	8.17415225126923e-001	8.17428194656259e-001	-1.2970e-005	8.17369828086118e-001	4.5397e-005
2.00	2.00	6.03500960611993e-001	6.03500960611993e-001	0.0000e+000	6.03500960611993e-001	0.0000e+000
2.00	3.00	4.14710585222029e-001	4.14710867477435e-001	-2.8226e-007	4.14828888409162e-001	-1.1830e-004
2.00	4.00	2.70039453942757e-001	2.69811654802219e-001	2.2780e-004	2.69621837766475e-001	4.1762e-004
2.00	5.00	1.68568913522749e-001	1.68497221983705e-001	7.1692e-005	1.68682419266487e-001	-1.1351e-004
3.00	1.00	9.06136886710340e-001	9.06267867579841e-001	-1.3098e-004	9.05542657196414e-001	5.9423e-004
3.00	2.00	7.53011300651773e-001	7.53013081872026e-001	-1.7812e-006	7.53236255746984e-001	-2.2496e-004
3.00	3.00	5.83328716319908e-001	5.83328716319908e-001	0.0000e+000	5.83328716319908e-001	0.0000e+000
3.00	4.00	4.26907556449231e-001	4.26918851025958e-001	-1.1295e-005	4.27066464775575e-001	-1.5891e-004
3.00	5.00	2.98193396308125e-001	2.98216836562828e-001	-2.3440e-005	2.98205531834536e-001	-1.2136e-005
4.00	1.00	9.52770303245878e-001	9.52815081514866e-001	-4.4778e-005	9.53343249350871e-001	-5.7295e-004
4.00	2.00	8.51936356981248e-001	8.51914807236771e-001	2.1550e-005	8.51935044764002e-001	1.3122e-006
4.00	3.00	7.16950482726697e-001	7.16954517666886e-001	-4.0349e-006	7.16818858591617e-001	1.3162e-004
4.00	4.00	5.71715890928425e-001	5.71715890928425e-001	0.0000e+000	5.71715890928425e-001	0.0000e+000
4.00	5.00	4.35072015844931e-001	4.35062221409677e-001	9.7944e-006	4.35115418770059e-001	-4.3403e-005
5.00	1.00	9.76650054658845e-001	9.76510807403662e-001	1.3925e-004	9.77406353687313e-001	-7.5630e-004
5.00	2.00	9.13934477595961e-001	9.14045763627111e-001	-1.1129e-004	9.13137511047949e-001	7.9697e-004
5.00	3.00	8.14938772496419e-001	8.14931554017805e-001	7.2185e-006	8.15157612873681e-001	-2.1884e-004
5.00	4.00	6.92981835299699e-001	6.92981895074277e-001	-5.9775e-008	6.92729664621689e-001	2.5217e-004
5.00	5.00	5.63916668581714e-001	5.63916668581714e-001	0.0000e+000	5.63916668581714e-001	0.0000e+000
1.00	1.00	6.54254161276836e-001	6.54254161276836e-001	0.0000e+000	6.54254161276836e-001	0.0000e+000
1.00	2.00	3.94296858899549e-001	3.94284283560691e-001	1.2575e-005	3.94289821520363e-001	7.0374e-006
1.00	3.00	2.24984708801310e-001	2.25106756223286e-001	-1.2205e-004	2.25201568381690e-001	-2.1686e-004
1.00	4.00	1.23381447904444e-001	1.23462951106194e-001	-8.1503e-005	1.23848846361134e-001	-4.6740e-004
1.00	5.00	6.56319493789875e-002	6.52106118200613e-002	4.2134e-004	6.60886489994547e-002	-4.5670e-004
2.00	1.00	8.17415225126923e-001	8.17417224502408e-001	-1.9994e-006	8.17368934652986e-001	4.6290e-005
2.00	2.00	6.03500960611993e-001	6.03500960611993e-001	0.0000e+000	6.03500960611993e-001	0.0000e+000
2.00	3.00	4.14710585222029e-001	4.14708811205497e-001	1.7740e-006	4.14685887842424e-001	2.4697e-005
2.00	4.00	2.70039453942757e-001	2.70010880914697e-001	2.8573e-005	2.70033737948816e-001	5.7160e-006
2.00	5.00	1.68568913522749e-001	1.68504439915073e-001	6.4474e-005	1.68762980877612e-001	-1.9407e-004
3.00	1.00	9.06136886710340e-001	9.06138733595694e-001	-1.8469e-006	9.06206888176947e-001	-7.0001e-005
3.00	2.00	7.53011300651773e-001	7.53011750858404e-001	-4.5021e-007	7.52948428029182e-001	6.2873e-005
3.00	3.00	5.83328716319908e-001	5.83328716319908e-001	0.0000e+000	5.83328716319908e-001	0.0000e+000
3.00	4.00	4.26907556449231e-001	4.26908169590950e-001	-6.1314e-007	4.26904211278509e-001	3.3452e-006
3.00	5.00	2.98193396308125e-001	2.98214992816497e-001	-2.1597e-005	2.98273515035349e-001	-8.0119e-005
4.00	1.00	9.52770303245878e-001	9.52771531277198e-001	-1.2280e-006	9.52748392634784e-001	2.1911e-005
4.00	2.00	8.51936356981248e-001	8.51917356942582e-001	1.9000e-005	8.52345858985399e-001	-4.0950e-004
4.00	3.00	7.16950482726697e-001	7.16951554059388e-001	-1.0713e-006	7.16958023384484e-001	-7.5407e-006
4.00	4.00	5.71715890928425e-001	5.71715890928425e-001	0.0000e+000	5.71715890928425e-001	0.0000e+000
4.00	5.00	4.35072015844931e-001	4.35073490478388e-001	-1.4746e-006	4.35075522776482e-001	-3.5069e-006
5.00	1.00	9.76650054658845e-001	9.76686506879731e-001	-3.6452e-005	9.76660803995853e-001	-1.0749e-005
5.00	2.00	9.13934477595961e-001	9.14013285590201e-001	-7.8808e-005	9.13516535189875e-001	4.1794e-004
5.00	3.00	8.14938772496419e-001	8.14935567195919e-001	3.2053e-006	8.15277495507355e-001	-3.3872e-004
5.00	4.00	6.92981835299699e-001	6.92982132104268e-001	-2.9680e-007	6.92958013262416e-001	2.3822e-005
5.00	5.00	5.63916668581714e-001	5.63916668581714e-001	0.0000e+000	5.63916668581714e-001	0.0000e+000

Table D.6: Estimates of $\rho(\sigma,\tau)$, based upon the estimators $\widehat{\Xi}_5$ and $\widehat{\Xi}_7$. For each (σ,τ) pair, an estimate is compared to one obtained by ASQ, with a tolerance of 10^{-8}. Both ϵ_1 and ϵ_2 are the absolute error between the ASQ estimate and $\widehat{\Xi}_5$ and $\widehat{\Xi}_7$ respectively. The first half of the table sets $N = 10^3$, second sets $N = 10^4$.

σ	τ	ASQ	$\widehat{\Xi}_5$	ϵ_1	$\widehat{\Xi}_7$	ϵ_2
1.00	1.00	6.54254161276836e-001	6.54254161276836e-001	0.0000e+000	6.54254161276836e-001	0.0000e+000
1.00	2.00	3.94296858899549e-001	3.94337020160272e-001	-4.0161e-005	3.95388002687352e-001	-1.0911e-003
1.00	3.00	2.24984708801310e-001	2.23575790417899e-001	1.4089e-003	2.30176619760505e-001	-5.1919e-003
1.00	4.00	1.23381447904444e-001	1.19938395744302e-001	3.4431e-003	1.22861809003696e-001	5.1964e-004
1.00	5.00	6.56319493789875e-002	6.10954220170256e-002	4.5365e-003	5.48755106171625e-002	1.0756e-002
2.00	1.00	8.17415225126923e-001	8.17489223701762e-001	-7.3999e-005	8.17498683336198e-001	-8.3458e-005
2.00	2.00	6.03500960611993e-001	6.03500960611993e-001	-1.1102e-016	6.03500960611993e-001	-1.1102e-016
2.00	3.00	4.14710585222029e-001	4.14740813509182e-001	-3.0228e-005	4.15934076372467e-001	-1.2235e-003
2.00	4.00	2.70039453942757e-001	2.69559062846534e-001	4.8039e-004	2.63295829783099e-001	6.7436e-003
2.00	5.00	1.68568913522749e-001	1.69860857919563e-001	-1.2919e-003	1.68819645820599e-001	-2.5073e-004
3.00	1.00	9.06136886710340e-001	9.01904787060865e-001	4.2321e-003	9.08129853290677e-001	-1.9930e-003
3.00	2.00	7.53011300651773e-001	7.52285085783106e-001	7.2621e-004	7.52267693086255e-001	7.4361e-004
3.00	3.00	5.83328716319908e-001	5.83328716319908e-001	0.0000e+000	5.83328716319908e-001	0.0000e+000
3.00	4.00	4.26907556449231e-001	4.26928778713931e-001	-2.1222e-005	4.26190593759042e-001	7.1696e-004
3.00	5.00	2.98193396308125e-001	2.98343281564006e-001	-1.4989e-004	3.01187332868557e-001	-2.9939e-003
4.00	1.00	9.52770303245878e-001	9.54695390161913e-001	-1.9251e-003	9.53726095138021e-001	-9.5579e-004
4.00	2.00	8.51936356981248e-001	8.51364177242597e-001	5.7218e-004	8.55724144726791e-001	-3.7878e-003
4.00	3.00	7.16950482726697e-001	7.16867988369372e-001	8.2494e-005	7.20623041903683e-001	-3.6726e-003
4.00	4.00	5.71715890928425e-001	5.71715890928425e-001	-1.1102e-016	5.71715890928425e-001	-1.1102e-016
4.00	5.00	4.35072015844931e-001	4.35084058735489e-001	-1.2043e-005	4.34085875608268e-001	9.8614e-004
5.00	1.00	9.76650054658845e-001	9.62599500259666e-001	1.4051e-002	9.72081476674733e-001	4.5686e-003
5.00	2.00	9.13934477595961e-001	9.09796433095005e-001	4.1380e-003	9.22999120866528e-001	-9.0646e-003
5.00	3.00	8.14938772496419e-001	8.15809512487437e-001	-8.7074e-004	8.22939825440480e-001	-8.0011e-003
5.00	4.00	6.92981835299699e-001	6.93022105037160e-001	-4.0270e-005	6.93273140069790e-001	-2.9130e-004
5.00	5.00	5.63916668581714e-001	5.63916668581714e-001	0.0000e+000	5.63916668581714e-001	0.0000e+000
1.00	1.00	6.54254161276836e-001	6.54254161276836e-001	0.0000e+000	6.54254161276836e-001	0.0000e+000
1.00	2.00	3.94296858899549e-001	3.94345228002909e-001	-4.8369e-005	3.93124862489673e-001	1.1720e-003
1.00	3.00	2.24984708801310e-001	2.25321566100041e-001	-3.3686e-004	2.23723283619554e-001	1.2614e-003
1.00	4.00	1.23381447904444e-001	1.23562762339163e-001	-1.8131e-004	1.32930198574313e-001	-9.5488e-003
1.00	5.00	6.56319493789875e-002	6.46436122153489e-002	9.8834e-004	7.31434113903418e-002	-7.5115e-003
2.00	1.00	8.17415225126923e-001	8.17169389301722e-001	2.4584e-004	8.17137884137984e-001	2.7734e-004
2.00	2.00	6.03500960611993e-001	6.03500960611993e-001	-1.1102e-016	6.03500960611993e-001	-1.1102e-016
2.00	3.00	4.14710585222029e-001	4.14700280601804e-001	1.0305e-005	4.14736182053095e-001	-2.5597e-005
2.00	4.00	2.70039453942757e-001	2.70135453609699e-001	-9.6000e-005	2.69782231562045e-001	2.5722e-004
2.00	5.00	1.68568913522749e-001	1.68163748884650e-001	4.0516e-004	1.77311558562444e-001	-8.7426e-003
3.00	1.00	9.06136886710340e-001	9.06296707904199e-001	-1.5982e-004	9.06457202559808e-001	-3.2032e-004
3.00	2.00	7.53011300651773e-001	7.52815329602791e-001	1.9597e-004	7.52935549034427e-001	7.5752e-005
3.00	3.00	5.83328716319908e-001	5.83328716319908e-001	0.0000e+000	5.83328716319908e-001	0.0000e+000
3.00	4.00	4.26907556449231e-001	4.26909740740315e-001	-2.1843e-006	4.26875420797241e-001	3.2136e-005
3.00	5.00	2.98193396308125e-001	2.98196704618595e-001	-3.3083e-006	2.98583224615187e-001	-3.8983e-004
4.00	1.00	9.52770303245878e-001	9.55673772718039e-001	-2.9035e-003	9.55385852690463e-001	-2.6155e-003
4.00	2.00	8.51936356981248e-001	8.51363278432103e-001	5.7308e-004	8.52108883024849e-001	-1.7253e-004
4.00	3.00	7.16950482726697e-001	7.16940864017119e-001	9.6187e-006	7.17242292980479e-001	-2.9181e-004
4.00	4.00	5.71715890928425e-001	5.71715890928425e-001	-1.1102e-016	5.71715890928425e-001	-1.1102e-016
4.00	5.00	4.35072015844931e-001	4.35063444041519e-001	8.5718e-006	4.35644991278726e-001	-5.7298e-004
5.00	1.00	9.76650054658845e-001	9.75668099368548e-001	9.8196e-004	9.87237539040348e-001	-1.0587e-002
5.00	2.00	9.13934477595961e-001	9.13998461328591e-001	-6.3984e-005	9.12912349040459e-001	1.0221e-003
5.00	3.00	8.14938772496419e-001	8.14969107463293e-001	-3.0335e-005	8.13606180455040e-001	1.3326e-003
5.00	4.00	6.92981835299699e-001	6.93012499430915e-001	-3.0664e-005	6.93111295941492e-001	-1.2946e-004
5.00	5.00	5.63916668581714e-001	5.63916668581714e-001	0.0000e+000	5.63916668581714e-001	0.0000e+000

Table D.7: A selection of estimates of $\rho(\sigma, \tau)$, based upon the estimators $\widehat{\Xi}_5$ and $\widehat{\Xi}_7$. For each (σ, τ) pair, an estimate is compared to one obtained by ASQ, with a tolerance of 10^{-8}. Both ϵ_1 and ϵ_2 are the absolute error between the ASQ estimate and $\widehat{\Xi}_5$ and $\widehat{\Xi}_7$ respectively. The first half of the table sets $N = 10^5$, second sets $N = 10^6$.

σ	τ	ASQ	$\widehat{\Xi}_5$	ϵ_1	$\widehat{\Xi}_7$	ϵ_2
1.00	1.00	6.54254161276836e-001	6.54254161276836e-001	0.0000e+000	6.54254161276836e-001	0.0000e+000
1.00	2.00	3.94296858899549e-001	3.94284164720893e-001	1.2694e-005	3.94281844400587e-001	1.5014e-005
1.00	3.00	2.24984708801310e-001	2.24908140763895e-001	7.6568e-005	2.24699192164545e-001	2.8552e-004
1.00	4.00	1.23381447904444e-001	1.23352781470946e-001	2.8666e-005	1.21023449364528e-001	2.3580e-003
1.00	5.00	6.56319493789875e-002	6.50613371093183e-002	5.7061e-004	6.15935083116151e-002	4.0384e-003
2.00	1.00	8.17415225126923e-001	8.17545275584011e-001	-1.3005e-004	8.17813260180413e-001	-3.9804e-004
2.00	2.00	6.03500960611993e-001	6.03500960611993e-001	-1.1102e-016	6.03500960611993e-001	-1.1102e-016
2.00	3.00	4.14710585222029e-001	4.14711620346561e-001	-1.0351e-006	4.14464332640283e-001	2.4625e-004
2.00	4.00	2.70039453942757e-001	2.70012728124271e-001	2.6726e-005	2.69848525144718e-001	1.9093e-004
2.00	5.00	1.68568913522749e-001	1.68813807613917e-001	-2.4489e-004	1.68419704034919e-001	1.4921e-004
3.00	1.00	9.06136886710340e-001	9.06587525200322e-001	-4.5064e-004	9.05588464872565e-001	5.4842e-004
3.00	2.00	7.53011300651773e-001	7.53008395831830e-001	2.9048e-006	7.52945102768093e-001	6.6198e-005
3.00	3.00	5.83328716319908e-001	5.83328716319908e-001	0.0000e+000	5.83328716319908e-001	0.0000e+000
3.00	4.00	4.26907556449231e-001	4.26908912710693e-001	-1.3563e-006	4.26878171154608e-001	2.9385e-005
3.00	5.00	2.98193396308125e-001	2.98173536434324e-001	1.9860e-005	2.98843492677548e-001	-6.5010e-004
4.00	1.00	9.52770303245878e-001	9.53060960489581e-001	-2.9066e-004	9.53296220751630e-001	-5.2592e-004
4.00	2.00	8.51936356981248e-001	8.52097010525492e-001	-1.6065e-004	8.51712638597254e-001	2.2372e-004
4.00	3.00	7.16950482726697e-001	7.16971463894443e-001	-2.0981e-005	7.17269308580283e-001	-3.1883e-004
4.00	4.00	5.71715890928425e-001	5.71715890928425e-001	-1.1102e-016	5.71715890928425e-001	-1.1102e-016
4.00	5.00	4.35072015844931e-001	4.35071937510675e-001	7.8334e-008	4.35075301709926e-001	-3.2859e-006
5.00	1.00	9.76650054658845e-001	9.75041885494213e-001	1.6082e-003	9.77518093870024e-001	-8.6804e-004
5.00	2.00	9.13934477595961e-001	9.14154412522131e-001	-2.1993e-004	9.14653733043966e-001	-7.1926e-004
5.00	3.00	8.14938772496419e-001	8.14979716879165e-001	-4.0944e-005	8.14696414119326e-001	2.4236e-004
5.00	4.00	6.92981835299699e-001	6.92986133787990e-001	-4.2985e-006	6.92988586091299e-001	-6.7508e-006
5.00	5.00	5.63916668581714e-001	5.63916668581714e-001	0.0000e+000	5.63916668581714e-001	0.0000e+000
1.00	1.00	6.54254161276836e-001	6.54254161276836e-001	0.0000e+000	6.54254161276836e-001	0.0000e+000
1.00	2.00	3.94296858899549e-001	3.94295900532927e-001	9.5837e-007	3.94263602876621e-001	3.3256e-005
1.00	3.00	2.24984708801310e-001	2.24955098437109e-001	2.9610e-005	2.25025141517534e-001	-4.0433e-005
1.00	4.00	1.23381447904444e-001	1.23483463203552e-001	-1.0202e-004	1.23452295130581e-001	-7.0847e-005
1.00	5.00	6.56319493789875e-002	6.55395465216552e-002	9.2403e-005	6.55852215586087e-002	4.6728e-005
2.00	1.00	8.17415225126923e-001	8.17420597856408e-001	-5.3727e-006	8.17387653324767e-001	2.7572e-005
2.00	2.00	6.03500960611993e-001	6.03500960611993e-001	-1.1102e-016	6.03500960611993e-001	-1.1102e-016
2.00	3.00	4.14710585222029e-001	4.14711569592460e-001	-9.8437e-007	4.14670516249809e-001	4.0069e-005
2.00	4.00	2.70039453942757e-001	2.70049465936515e-001	-1.0012e-005	2.70358553038496e-001	-3.1910e-004
2.00	5.00	1.68568913522749e-001	1.68625906091264e-001	-5.6993e-005	1.69106112680680e-001	-5.3720e-004
3.00	1.00	9.06136886710340e-001	9.06023453255840e-001	1.1343e-004	9.06298843393719e-001	-1.6196e-004
3.00	2.00	7.53011300651773e-001	7.53018907408069e-001	-7.6068e-006	7.53007379700900e-001	3.9210e-006
3.00	3.00	5.83328716319908e-001	5.83328716319908e-001	0.0000e+000	5.83328716319908e-001	0.0000e+000
3.00	4.00	4.26907556449231e-001	4.26908022888785e-001	-4.6644e-007	4.26880294621044e-001	2.7262e-005
3.00	5.00	2.98193396308125e-001	2.98187405379171e-001	5.9909e-006	2.98244736065811e-001	-5.1340e-005
4.00	1.00	9.52770303245878e-001	9.53023383446578e-001	-2.5308e-004	9.53195095899797e-001	-4.2479e-004
4.00	2.00	8.51936356981248e-001	8.51904739045078e-001	3.1618e-005	8.51756598434958e-001	1.7976e-004
4.00	3.00	7.16950482726697e-001	7.16950460863216e-001	2.1863e-008	7.16956805796781e-001	-6.3231e-006
4.00	4.00	5.71715890928425e-001	5.71715890928425e-001	-1.1102e-016	5.71715890928425e-001	-1.1102e-016
4.00	5.00	4.35072015844931e-001	4.35072248098194e-001	-2.3225e-007	4.35032701775758e-001	3.9314e-005
5.00	1.00	9.76650054658845e-001	9.76322360488141e-001	3.2769e-004	9.76595893987970e-001	5.4161e-005
5.00	2.00	9.13934477595961e-001	9.14025655292749e-001	-9.1178e-005	9.14137388458337e-001	-2.0291e-004
5.00	3.00	8.14938772496419e-001	8.14927801855803e-001	1.0971e-005	8.14975773039055e-001	-3.7001e-005
5.00	4.00	6.92981835299699e-001	6.92980018791908e-001	1.8165e-006	6.92976000416266e-001	5.8349e-006
5.00	5.00	5.63916668581714e-001	5.63916668581714e-001	0.0000e+000	5.63916668581714e-001	0.0000e+000

Table D.8: The first half of the table is based upon the Uniform estimators of Theorem 1 Part (iii), $\widehat{\Xi}_4$ and Theorem 1 Part (iv), $\widehat{\Xi}_5$. The second half of the table is based upon the exponential estimators of Theorem 1 Part (iii), $\widehat{\Xi}_6$ and Theorem 1 Part (iv), $\widehat{\Xi}_7$, giving a selection of estimates of $\rho(\sigma,\tau)$, $N = 10^6$. For each (σ,τ) pair, an estimate is compared to one obtained by ASQ, with a tolerance of 10^{-8}.

σ	τ	ASQ	$\widehat{\Xi}_4$	ϵ_1	$\widehat{\Xi}_5$	ϵ_2
1.00	1.00	6.54254161276836e-001	6.54254161276836e-001	0.0000e+000	6.54254161276836e-001	0.0000e+000
1.00	2.00	3.94296858899549e-001	3.94307315651829e-001	-1.0457e-005	3.94297970230268e-001	-1.1113e-006
1.00	3.00	2.24984708801310e-001	2.24866226216644e-001	1.1848e-004	2.25009057360205e-001	-2.4349e-005
1.00	4.00	1.23381447904444e-001	1.23453201191199e-001	-7.1753e-005	1.23356589294396e-001	2.4859e-005
1.00	5.00	6.56319493789875e-002	6.55507947563719e-002	8.1155e-005	6.56130051310764e-002	1.8944e-005
2.00	1.00	8.17415225126923e-001	8.17415581500774e-001	-3.5637e-007	8.17357071863731e-001	5.8153e-005
2.00	2.00	6.03500960611993e-001	6.03500960611993e-001	0.0000e+000	6.03500960611993e-001	-1.1102e-016
2.00	3.00	4.14710585222029e-001	4.14722245366149e-001	-1.1660e-005	4.14708135723465e-001	2.4495e-006
2.00	4.00	2.70039453942757e-001	2.70046023461417e-001	-6.5695e-006	2.70026773998732e-001	1.2680e-005
2.00	5.00	1.68568913522749e-001	1.68397961576931e-001	1.7095e-004	1.68570649199877e-001	-1.7357e-006
3.00	1.00	9.06136886710340e-001	9.06113730157681e-001	2.3157e-005	9.06119546174549e-001	1.7341e-005
3.00	2.00	7.53011300651773e-001	7.53011556567329e-001	-2.5592e-007	7.53016352330771e-001	-5.0517e-006
3.00	3.00	5.83328716319908e-001	5.83328716319908e-001	0.0000e+000	5.83328716319908e-001	0.0000e+000
3.00	4.00	4.26907556449231e-001	4.26897091677684e-001	1.0465e-005	4.26908828342268e-001	-1.2719e-006
3.00	5.00	2.98193396308125e-001	2.98226020809542e-001	-3.2625e-005	2.98187454122406e-001	5.9422e-006
4.00	1.00	9.52770303245878e-001	9.52718255950275e-001	5.2047e-005	9.52710617939466e-001	5.9685e-005
4.00	2.00	8.51936356981248e-001	8.51916535861328e-001	1.9821e-005	8.51915813001954e-001	2.0544e-005
4.00	3.00	7.16950482726697e-001	7.16952035552288e-001	-1.5528e-006	7.16950131322788e-001	3.5140e-007
4.00	4.00	5.71715890928425e-001	5.71715890928425e-001	0.0000e+000	5.71715890928425e-001	-1.1102e-016
4.00	5.00	4.35072015844931e-001	4.35071476520327e-001	5.3932e-007	4.35071086007455e-001	9.2984e-007
5.00	1.00	9.76650054658845e-001	9.76578388776317e-001	7.1666e-005	9.76614088274783e-001	3.5966e-005
5.00	2.00	9.13934477595961e-001	9.13959287867896e-001	-2.4810e-005	9.13833651392929e-001	1.0083e-004
5.00	3.00	8.14938772496419e-001	8.14934020801279e-001	4.7517e-006	8.14968562488102e-001	-2.9790e-005
5.00	4.00	6.92981835299699e-001	6.92981382024522e-001	4.5328e-007	6.92985541326024e-001	-3.7060e-006
5.00	5.00	5.63916668581714e-001	5.63916668581714e-001	0.0000e+000	5.63916668581714e-001	0.0000e+000

σ	τ	ASQ	$\widehat{\Xi}_6$	ϵ_3	$\widehat{\Xi}_7$	ϵ_4
1.00	1.00	6.54254161276836e-001	6.54254161276836e-001	0.0000e+000	6.54254161276836e-001	0.0000e+000
1.00	2.00	3.94296858899549e-001	3.94293172340004e-001	3.6866e-006	3.94290150537489e-001	6.7084e-006
1.00	3.00	2.24984708801310e-001	2.24985701862081e-001	-9.9306e-007	2.24960781602674e-001	2.3927e-005
1.00	4.00	1.23381447904444e-001	1.23735622359681e-001	-3.5417e-004	1.23121204232793e-001	2.6024e-004
1.00	5.00	6.56319493789875e-002	6.56295431629140e-002	2.4062e-006	6.62014160940180e-002	-5.6947e-004
2.00	1.00	8.17415225126923e-001	8.17291464590203e-001	1.2376e-004	8.17456637111941e-001	-4.1412e-005
2.00	2.00	6.03500960611993e-001	6.03500960611993e-001	0.0000e+000	6.03500960611993e-001	-1.1102e-016
2.00	3.00	4.14710585222029e-001	4.14696298790895e-001	1.4286e-005	4.14658657938413e-001	5.1927e-005
2.00	4.00	2.70039453942757e-001	2.70330844666473e-001	-2.9139e-004	2.70094136439180e-001	-5.4682e-005
2.00	5.00	1.68568913522749e-001	1.68618181586572e-001	-4.9268e-005	1.68621722495285e-001	-5.2809e-005
3.00	1.00	9.06136886710340e-001	9.06306590322955e-001	-1.6970e-004	9.05945882452053e-001	1.9100e-004
3.00	2.00	7.53011300651773e-001	7.52974110707278e-001	3.7190e-005	7.52974806352080e-001	3.6494e-005
3.00	3.00	5.83328716319908e-001	5.83328716319908e-001	0.0000e+000	5.83328716319908e-001	0.0000e+000
3.00	4.00	4.26907556449231e-001	4.26907146953775e-001	4.0950e-007	4.26942387727101e-001	-3.4831e-005
3.00	5.00	2.98193396308125e-001	2.98159249548400e-001	3.4147e-005	2.98131931054534e-001	6.1465e-005
4.00	1.00	9.52770303245878e-001	9.52817097645603e-001	-4.6794e-005	9.52900687884700e-001	-1.3038e-004
4.00	2.00	8.51936356981248e-001	8.51859884522794e-001	7.6472e-005	8.51767118116811e-001	1.6924e-004
4.00	3.00	7.16950482726697e-001	7.16997642114752e-001	-4.7159e-005	7.16959665066951e-001	-9.1823e-006
4.00	4.00	5.71715890928425e-001	5.71715890928425e-001	0.0000e+000	5.71715890928425e-001	-1.1102e-016
4.00	5.00	4.35072015844931e-001	4.35032529750453e-001	3.9486e-005	4.35090931019471e-001	-1.8915e-005
5.00	1.00	9.76650054658845e-001	9.76734872444844e-001	-8.4818e-005	9.76133843741234e-001	5.1621e-004
5.00	2.00	9.13934477595961e-001	9.13868262110131e-001	6.6215e-005	9.13908669263083e-001	2.5808e-005
5.00	3.00	8.14938772496419e-001	8.14870703703569e-001	6.8069e-005	8.14982387717804e-001	-4.3615e-005
5.00	4.00	6.92981835299699e-001	6.92975020678868e-001	6.8146e-006	6.93007914036494e-001	-2.6079e-005
5.00	5.00	5.63916668581714e-001	5.63916668581714e-001	0.0000e+000	5.63916668581714e-001	0.0000e+000

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19. ABSTRACT

The Marcum Q-Function is an important tool in the study of radar detection probabilities in Gaussian clutter and noise. Due to the fact that it is an intractable integral, much research has focused on finding good numerical approximations for it. Such approximations include numerical integration techniques, such as adaptive Simpson quadrature, and Taylor series approximations, induced by the modified Bessel function of order zero, which appears in the integrand. One technique which has not been explored in the literature is the sampling-based Monte Carlo approach. Part of the reason for this is that the integral representation of the Marcum Q-Function is not in the most suitable form for Monte Carlo methods. Using some recently derived techniques, we construct a number of sampling-based estimators of this function, and we consider their relative merits.